Lecture 4: NHDM with non-abelian symmetries

Igor Ivanov

IFPA, University of Liège, Belgium
Institute of Mathematics, Novosibirsk, Russia

DCPIHEP workshop, Colima, Mexico, January 6-16, 2014

Outline

Abelian “LEGO”

In Lecture 3, we saw a method to systematically detect all abelian groups of any bSM model. Now we want to use these “building blocks” to construct various non-abelian groups possible in a given model — sort of a “LEGO” of abelian groups.

Abelian "LEGO"

In Lecture 3, we saw a method to systematically detect all abelian groups of any bSM model. Now we want to use these "building blocks" to construct various non-abelian groups possible in a given model — sort of a "LEGO" of abelian groups.

Again, I stress that we focus only on those groups which represent the genuine symmetry content of a model. This is especially true for finite groups: we want to find such groups which, when imposed on a model, do not automatically lead to continuous symmetries.

Abelian "LEGO"

In Lecture 3, we saw a method to systematically detect all abelian groups of any bSM model. Now we want to use these "building blocks" to construct various non-abelian groups possible in a given model — sort of a "LEGO" of abelian groups.

Again, I stress that we focus only on those groups which represent the genuine symmetry content of a model. This is especially true for finite groups: we want to find such groups which, when imposed on a model, do not automatically lead to continuous symmetries.

The method which I will show is, in principle, applicable to any bSM model. However, it is rather difficult to go up to the end. The only example so far where it was successfully applied and led to non-trivial results is the scalar sector of 3HDM *[Ivanov, Vdovin, 2013]*.

Abelian "LEGO"

Suppose we have a finite non-abelian symmetry group G of NHDM scalar sector. G can be described by generators $a_1, a_2, a_3, \ldots,$ not necessarily commuting, which satisfy certain relations.

Abelian "LEGO"

Suppose we have a finite non-abelian symmetry group G of NHDM scalar sector. G can be described by generators $a_1, a_2, a_3, \dots,$ not necessarily commuting, which satisfy certain relations.

Each generator can be used to define a cyclic subgroup. For example, if $a_1^{n_1} = e$, then

$$A_1 = \{e, a_1, a_1^2, \dots, a_1^{n_1-1}\} \simeq \mathbb{Z}_{n_1} .$$

Similarly, we get build cyclic subgroups A_2, A_3, and possibly non-cyclic abelian subgroups.

Abelian “LEGO”

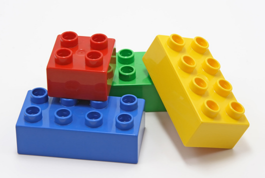
Suppose we have a finite non-abelian symmetry group G of NHDM scalar sector. G can be described by generators $a_1, a_2, a_3, \dots,$ not necessarily commuting, which satisfy certain relations.

Each generator can be used to define a cyclic subgroup. For example, if $a_1^{n_1} = e$, then

$$A_1 = \{e, a_1, a_1^2, \dots, a_1^{n_1-1}\} \simeq \mathbb{Z}_{n_1} \, .$$

Similarly, we get build cyclic subgroups A_2, A_3, and possibly non-cyclic abelian subgroups.

Key step 1: if we have the full list of possible abelian groups a given model, then any possible G can have only those groups and nothing else. These are “LEGO bricks” with which we will build a non-abelian model.



Abelian "LEGO"

In Lecture 3, we studied rephasing groups. Now, abelian groups don't have to represent rephasing!

Abelian "LEGO"

In Lecture 3, we studied rephasing groups. Now, abelian groups don't have to represent rephasing! For example, in 3HDM we can easily have a $\mathbb{Z}_3$ group generated by

$$a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \text{ with } \omega \equiv \exp\left(\frac{2\pi i}{3}\right), \quad \text{or } a_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We just need to remember that a_2 can be mapped to a rephasing transformation under certain basis change, but we do not require that both a_1 and a_2 become rephasing transformations simultaneously.

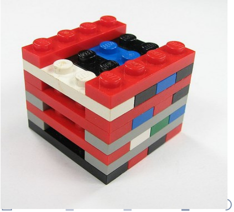
Abelian "LEGO"

In Lecture 3, we studied rephasing groups. Now, abelian groups don't have to represent rephasing! For example, in 3HDM we can easily have a $\mathbb{Z}_3$ group generated by

$$a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \text{ with } \omega \equiv \exp\left(\frac{2\pi i}{3}\right), \quad \text{or } a_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We just need to remember that a_2 can be mapped to a rephasing transformation under certain basis change, but we do not require that both a_1 and a_2 become rephasing transformations simultaneously.

So, various abelian subgroups A_1, A_2, ..., can be "oriented" differently inside G. That's fine, we just need to take it into account when building group G.



Abelian “LEGO”

Note that you cannot combine “abelian bricks” in arbitrary way! You need to make checks.

Sometimes you can get groups G with extra abelian subgroups, absent from the list of A_i $\rightarrow$ these G's are disregarded.

Abelian "LEGO"

Note that you cannot combine "abelian bricks" in arbitrary way! You need to make checks.

Sometimes you can get groups G with extra abelian subgroups, absent from the list of A_i $\to$ these G's are disregarded.

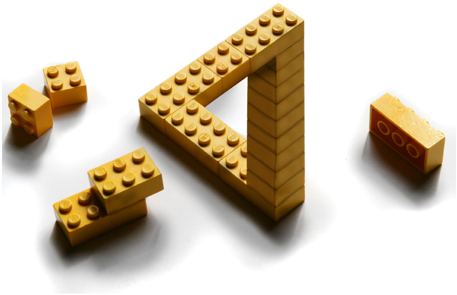
Some groups might look OK but they just don't fit the space of N Higgs doublets.

Abelian “LEGO”

Note that you cannot combine “abelian bricks” in arbitrary way! You need to make checks.

Sometimes you can get groups G with extra abelian subgroups, absent from the list of $A_i \to$ these G’s are disregarded.

Some groups might look OK but they just don’t fit the space of N Higgs doublets.



Non-abelian groups in 3HDM

So, skipping all details, in *[Ivanov, Vdovin, 2013]* we made key step 2: we proved that any finite group G in scalar sector of 3HDM must have a invariant maximal abelian subgroup A.

Non-abelian groups in 3HDM

So, skipping all details, in *[Ivanov, Vdovin, 2013]* we made key step 2: we proved that any finite group G in scalar sector of 3HDM must have a invariant maximal abelian subgroup A.

This means that A combines two properties:

- it does not lie in a larger abelian subgroup of G $\to$ nothing else in G commutes with the entire A,
- it is invariant in G: $g^{-1}Ag = A \;\; \forall g \in G$.

Non-abelian groups in 3HDM

So, skipping all details, in *[Ivanov, Vdovin, 2013]* we made key step 2: we proved that any finite group G in scalar sector of 3HDM must have a invariant maximal abelian subgroup A.

This means that A combines two properties:

- it does not lie in a larger abelian subgroup of G $\to$ nothing else in G commutes with the entire A,
- it is invariant in G: $g^{-1}Ag = A \quad \forall g \in G$.

This group-theoretic knowledge is HUGELY important, as it immediately gives the structure of any G:

Key step 3

$$G \simeq A \rtimes K\,, \quad \text{where } K \subseteq Aut(A)\,.$$

Non-abelian groups in 3HDM

A is invariant, $g^{-1}Ag = A \ \ \forall g \in G$, but it does not mean that each element $a \in A$ is invariant under this action!

Each element g induces a permutation on A:

$$a_i \mapsto g^{-1} a_i g = a_j \,.$$

This permutation is called an automorphism of A.

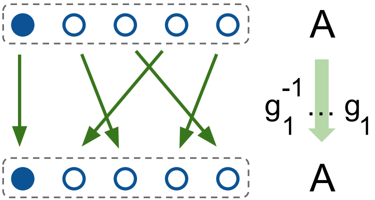
Non-abelian groups in 3HDM

A is invariant, $g^{-1}Ag = A \;\; \forall g \in G$, but it does not mean that each element $a \in A$ is invariant under this action!

Each element g induces a permutation on A:

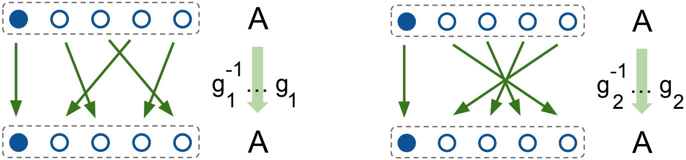
$$a_i \mapsto g^{-1} a_i g = a_j \,.$$

This permutation is called an automorphism of A.



Non-abelian groups in 3HDM

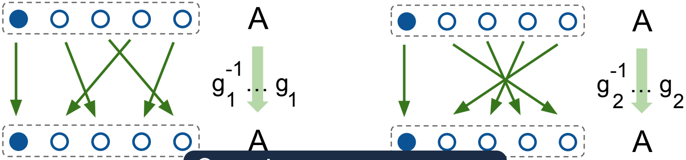
The fact that A is maximal abelian subgroup of G means that essentially different g's induce different automorphisms on A:



Here, "essentially different" means $g_1 \neq a \cdot g_2$.

Non-abelian groups in 3HDM

The fact that A is maximal abelian subgroup of G means that essentially different g's induce different automorphisms on A:



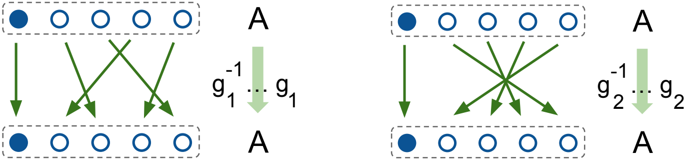
Here, "essentially different

Question

Q4.1: prove it!

Non-abelian groups in 3HDM

The fact that A is maximal abelian subgroup of G means that essentially different g's induce different automorphisms on A:

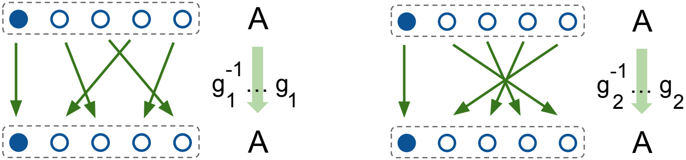


Here, "essentially different" means $g_1 \neq a \cdot g_2$.

All automorphisms of A form the automorphism group, *Aut A*.

Non-abelian groups in 3HDM

The fact that A is maximal abelian subgroup of G means that essentially different g's induce different automorphisms on A:

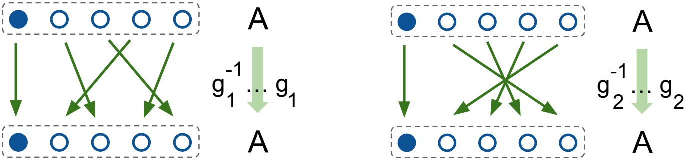


Here, "essentially different" means $g_1 \neq a \cdot g_2$.

All automorphisms of A form the automorphism group, $Aut\,A$.
Some of them are induced by various essentially different g's.
We conclude that $G/A = K\,,\ \ K \subseteq Aut\,A$.

Non-abelian groups in 3HDM

The fact that A is maximal abelian subgroup of G means that essentially different g's induce different automorphisms on A:



Here, "essentially different" means $g_1 \neq a \cdot g_2$.

All automorphisms of A form the automorphism group, $Aut\,A$.
Some of them are induced by various essentially different g's.
We conclude that $G/A = K\,,\ \ K \subseteq Aut\,A$.

This leads to construction of G as extension of A by K: $G = A \rtimes K$.

Non-abelian groups in 3HDM

In the scalar sector of 3HDM, we have the following abelian groups available:

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_3 \times \mathbb{Z}_3 \,.$$

So, we just need to calculate $Aut\, A$ for each of them, find all subgroups $K \subset Aut\, A$, then for each A and K, construct all $A \rtimes K$, and finally check if the result can fit inside 3HDM.

Non-abelian groups in 3HDM

In the scalar sector of 3HDM, we have the following abelian groups available:

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_3 \times \mathbb{Z}_3\,.$$

So, we just need to calculate $Aut\, A$ for each of them, find all subgroups $K \subset Aut\, A$, then for each A and K, construct all $A \rtimes K$, and finally check if the result can fit inside 3HDM. Results for non-abelian G:

$$S_3\,, \quad D_4\,, \quad A_4\,, \quad S_4\,, \quad \Delta(54)/\mathbb{Z}_3\,, \quad \Sigma(36)\,.$$

This list is complete: trying to impose any other finite Higgs-family symmetry group on the 3HDM potential will lead to a potential symmetric under a continuous group.

Non-abelian groups in 3HDM

A short example: suppose $A = \mathbb{Z}_4 = \{e, a, a^2, a^3\}$.

Then $Aut\, A = \mathbb{Z}_2$, it contains the trivial automorphism $a \mapsto a$ and one non-trivial automorphism $a \mapsto a^3$.

Non-abelian groups in 3HDM

A short example: suppose $A = \mathbb{Z}_4 = \{e, a, a^2, a^3\}$.

Then $Aut\,A = \mathbb{Z}_2$, it contains the trivial automorphism $a \mapsto a$ and one non-trivial automorphism $a \mapsto a^3$.

The semidirect product $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$ can lead to four groups:

- $\mathbb{Z}_4 \times \mathbb{Z}_2$ — abelian, absent from the list → disregard;

Non-abelian groups in 3HDM

A short example: suppose $A = \mathbb{Z}_4 = \{e, a, a^2, a^3\}$.

Then $Aut\, A = \mathbb{Z}_2$, it contains the trivial automorphism $a \mapsto a$ and one non-trivial automorphism $a \mapsto a^3$.

The semidirect product $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$ can lead to four groups:

- $\mathbb{Z}_4 \times \mathbb{Z}_2$ — abelian, absent from the list $\to$ disregard;
- $\mathbb{Z}_8$ — abelian, absent from the list $\to$ disregard;

Non-abelian groups in 3HDM

A short example: suppose $A = \mathbb{Z}_4 = \{e, a, a^2, a^3\}$.

Then $Aut\,A = \mathbb{Z}_2$, it contains the trivial automorphism $a \mapsto a$ and one non-trivial automorphism $a \mapsto a^3$.

The semidirect product $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$ can lead to four groups:

- $\mathbb{Z}_4 \times \mathbb{Z}_2$ — abelian, absent from the list $\to$ disregard;
- $\mathbb{Z}_8$ — abelian, absent from the list $\to$ disregard;
- D_4, symmetry group of the square — non-abelian, fits 3HDM, does not lead to any higher group $\to$ keep;

Non-abelian groups in 3HDM

A short example: suppose $A = \mathbb{Z}_4 = \{e, a, a^2, a^3\}$.

Then $Aut\, A = \mathbb{Z}_2$, it contains the trivial automorphism $a \mapsto a$ and one non-trivial automorphism $a \mapsto a^3$.

The semidirect product $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$ can lead to four groups:

- $\mathbb{Z}_4 \times \mathbb{Z}_2$ — abelian, absent from the list $\to$ disregard;
- $\mathbb{Z}_8$ — abelian, absent from the list $\to$ disregard;
- D_4, symmetry group of the square — non-abelian, fits 3HDM, does not lead to any higher group $\to$ keep;
- Q_4, quaternion group — non-abelian, fits 3HDM, but the potential is automatically symmetric under a continuous group $\to$ disregard.

EWSB in NHDM

The vacuum state is determined by the minimum of the Higgs potential $\to$ doublets ϕ_i acquire some vevs $\langle\phi_i\rangle$.
As in 2HDM, we can have three sorts of electroweak symmetry breaking:

- Electroweak vacuum: $\langle\phi_i\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,, \quad i = 1, \dots, N\,.$
- Neutral vacuum: $\langle\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_1 \end{pmatrix}\,,\ \langle\phi_i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ |v_i|e^{i\xi_i} \end{pmatrix},\ i \geq 2.$
  The space of neutral vacua is parametrized with $2N-1$ parameters.
- Charge-breaking vacuum:

$$\langle\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_1 \end{pmatrix}\,,\ \langle\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} u_2 \\ |v_2|e^{i\xi_2} \end{pmatrix}\,,\ \langle\phi_i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} |u_i|e^{i\zeta_i} \\ |v_i|e^{i\xi_i} \end{pmatrix}\,,\ i \geq 3\,.$$

  The space of charge-breaking vacua is parametrized with $4N-4$ parameters.

EWSB in NHDM

The vacuum state is determined by the minimum of the Higgs potential $\to$ doublets ϕ_i acquire some vevs $\langle\phi_i\rangle$.
As in 2HDM, we can have three sorts of electroweak symmetry breaking:

- Electroweak vacuum: $\langle\phi_i\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,, \quad i = 1, \dots, N\,.$
- Neutral vacuum: $\langle\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_1 \end{pmatrix}\,,\ \langle\phi_i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ |v_i|e^{i\xi_i} \end{pmatrix},\ i \geq 2.$
  The space of neutral vacua is parametrized with $2N-1$ parameters.
- Charge-breaking vacuum:

  The space of charge-breaking vacua is parametrized with $4N-4$ parameters.

We will work only with neutral vacua.

Types of symmetry breaking in scalar sector

Suppose the Higgs potential of NHDM is invariant under group G. After EWSB, we get a neutral vacuum with vev alignment v_i, which is invariant under vacuum symmetry group G_v.

Types of symmetry breaking in scalar sector

Suppose the Higgs potential of NHDM is invariant under group G. After EWSB, we get a neutral vacuum with vev alignment v_i, which is invariant under vacuum symmetry group G_v.

There can be three possibilities:

- symmetry is conserved: $G_v = G$;
- symmetry is partially broken: $\{e\} \subset G_v \subset G$;
- symmetry is completely broken: $G_v = \{e\}$.

Phenomenology depends on how much of the original symmetry is broken!

Types of symmetry breaking in scalar sector

Suppose the Higgs potential of NHDM is invariant under group G. After EWSB, we get a neutral vacuum with vev alignment v_i, which is invariant under vacuum symmetry group G_v.

There can be three possibilities:

- symmetry is conserved: $G_v = G$;
- symmetry is partially broken: $\{e\} \subset G_v \subset G$;
- symmetry is completely broken: $G_v = \{e\}$.

Phenomenology depends on how much of the original symmetry is broken!

If we already know the global minimum of a given NHDM potential, we can find G_v. But can we deduce something general?

Breaking of large symmetry groups

- Sufficiently large groups must break. For example, if ϕ's are in irrep of G, then G cannot be conserved in the vacuum.

Breaking of large symmetry groups

- Sufficiently large groups must break. For example, if ϕ's are in irrep of G, then G cannot be conserved in the vacuum.

  Suppose vev alignment v_i is invariant under G. Then, the 1D subspace defined by v_i is invariant under G, and so is its orthogonal complement. But it means that ϕ_i are in a reducible representation of G.

Breaking of large symmetry groups

- Sufficiently large groups must break. For example, if ϕ's are in irrep of G, then G cannot be conserved in the vacuum.

  Suppose vev alignment v_i is invariant under G. Then, the 1D subspace defined by v_i is invariant under G, and so is its orthogonal complement. But it means that ϕ_i are in a reducible representation of G.

- Sufficiently large groups cannot break completely.

Breaking of large symmetry groups

- Sufficiently large groups must break. For example, if ϕ's are in irrep of G, then G cannot be conserved in the vacuum.

  Suppose vev alignment v_i is invariant under G. Then, the 1D subspace defined by v_i is invariant under G, and so is its orthogonal complement. But it means that ϕ_i are in a reducible representation of G.

- Sufficiently large groups cannot break completely.

  Not yet been accurately proven! An idea of proof: complete breaking of large finite group G would produce very many degenerate global minima.

Question

Q4.2: prove that it produces at least $|G|$ degenerate global minima.

Breaking of large symmetry groups

- Sufficiently large groups must break. For example, if ϕ's are in irrep of G, then G cannot be conserved in the vacuum.

  Suppose vev alignment v_i is invariant under G. Then, the 1D subspace defined by v_i is invariant under G, and so is its orthogonal complement. But it means that ϕ_i are in a reducible representation of G.

- Sufficiently large groups cannot break completely.

  Not yet been accurately proven! An idea of proof: complete breaking of large finite group G would produce very many degenerate global minima. But the potential is a very simple function of fields: polynomial of degree 4.

Breaking of large symmetry groups

- Sufficiently large groups must break. For example, if ϕ's are in irrep of G, then G cannot be conserved in the vacuum.

  Suppose vev alignment v_i is invariant under G. Then, the 1D subspace defined by v_i is invariant under G, and so is its orthogonal complement. But it means that ϕ_i are in a reducible representation of G.

- Sufficiently large groups cannot break completely.

  Not yet been accurately proven! An idea of proof: complete breaking of large finite group G would produce very many degenerate global minima. But the potential is a very simple function of fields: polynomial of degree 4. There exists an upper bound n_{max} to the number of its minima. Therefore, of $|G| > n_{max}$, the group G cannot break completely.

Breaking of large symmetry groups

- Sufficiently large groups must break. For example, if ϕ's are in irrep of G, then G cannot be conserved in the vacuum.

  Suppose vev alignment v_i is invariant under G. Then, the 1D subspace defined by v_i is invariant under G, and so is its orthogonal complement. But it means that ϕ_i are in a reducible representation of G.

- Sufficiently large groups cannot break completely.

  Not yet been accurately proven! An idea of proof: complete breaking of large finite group G would produce very many degenerate global minima. But the potential is a very simple function of fields: polynomial of degree 4. There exists an upper bound n_{max} to the number of its minima. Therefore, of $|G| > n_{max}$, the group G cannot break completely.

  Unfortunately, it is difficult to find the exact upper bound, so it is difficult to make this statement more precise.

Breaking of large symmetry groups

So, if G is a sufficienty large finite group, the vacuum carries some remnants of G: $\{e\} \subset G_v \subset G$.

Many specific questions arise:

- what does "sufficienty large" mean exactly?
- how large and small can G_v be?
- can minima with different breaking patterns of G coexist?

Breaking of large symmetry groups

So, if G is a sufficienty large finite group, the vacuum carries some remnants of G: $\{e\} \subset G_v \subset G$.

Many specific questions arise:

- what does "sufficienty large" mean exactly?
- how large and small can G_v be?
- can minima with different breaking patterns of G coexist?

These are still open questions!

Minimization of potentials

Even in 2HDM, the general scalar potential cannot be minimized algebraically. If a symmetry is present, it reduces the number of free parameters and simplfies structures $\to$ minimization can be done manually. The fewer free parameters we have, the simpler is minimization.

Minimization of potentials

Even in 2HDM, the general scalar potential cannot be minimized algebraically. If a symmetry is present, it reduces the number of free parameters and simplfies structures → minimization can be done manually. The fewer free parameters we have, the simpler is minimization.

Usual and boresome procedure for minimization:

- solve coupled system: $\partial V/\partial\phi_i = 0$;
- compute $\partial^2 V/\partial\phi_i\partial\phi_j$, check that it is positive semdefinite;
- find all such points and select the deepest one.

Minimization of potentials

Even in 2HDM, the general scalar potential cannot be minimized algebraically. If a symmetry is present, it reduces the number of free parameters and simplfies structures $\to$ minimization can be done manually. The fewer free parameters we have, the simpler is minimization.

Usual and boresome procedure for minimization:

- solve coupled system: $\partial V/\partial\phi_i = 0$;
- compute $\partial^2 V/\partial\phi_i\partial\phi_j$, check that it is positive semdefinite;
- find all such points and select the deepest one.

For large symmetries with very few parameters, this is not needed! There is a simple method with immediately gives you the global minimum without derivatives.

$\Sigma(36)$ 3HDM

Consider a specific example: $\Sigma(36)$-symmetric 3HDM.

$$\begin{aligned} V &= -m^2\left(\phi_1^\dagger\phi_1+\phi_2^\dagger\phi_2+\phi_3^\dagger\phi_3\right)+\lambda_0\left(\phi_1^\dagger\phi_1+\phi_2^\dagger\phi_2+\phi_3^\dagger\phi_3\right)^2 \\ &\quad +\lambda_1\Big\{(\phi_1^\dagger\phi_1)^2+(\phi_2^\dagger\phi_2)^2+(\phi_3^\dagger\phi_3)^2-(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)-(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)-(\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1) \\ &\qquad\qquad +3(|\phi_1^\dagger\phi_2|^2+|\phi_2^\dagger\phi_3|^2+|\phi_3^\dagger\phi_1|^2)\Big\} \\ &\quad +\lambda_2\left(|\phi_1^\dagger\phi_2-\phi_2^\dagger\phi_3|^2+|\phi_2^\dagger\phi_3-\phi_3^\dagger\phi_1|^2+|\phi_3^\dagger\phi_1-\phi_1^\dagger\phi_2|^2\right). \end{aligned}$$

It is symmetric under the group called $\Sigma(36)$ generated by

$$a=\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix},\quad d=\frac{i}{\sqrt{3}}\begin{pmatrix}1&1&1\\1&\omega^2&\omega\\1&\omega&\omega^2\end{pmatrix},\quad \omega\equiv\exp\left(\frac{2\pi i}{3}\right),$$

and arbitrary permutations of doublets.

$\Sigma(36)$ 3HDM

Consider a specific example: $\Sigma(36)$-symmetric 3HDM.

$$\begin{aligned} V \;=\; & -m^2\left(\phi_1^\dagger\phi_1+\phi_2^\dagger\phi_2+\phi_3^\dagger\phi_3\right)+\lambda_0\left(\phi_1^\dagger\phi_1+\phi_2^\dagger\phi_2+\phi_3^\dagger\phi_3\right)^2 \\ & +\lambda_1\Big\{(\phi_1^\dagger\phi_1)^2+(\phi_2^\dagger\phi_2)^2+(\phi_3^\dagger\phi_3)^2-(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)-(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)-(\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1) \\ & \qquad +3(|\phi_1^\dagger\phi_2|^2+|\phi_2^\dagger\phi_3|^2+|\phi_3^\dagger\phi_1|^2)\Big\} \\ & +\lambda_2\left(|\phi_1^\dagger\phi_2-\phi_2^\dagger\phi_3|^2 \;[illegible]\; |^2\right)\,. \end{aligned}$$

Question

Q4.3: check it!

It is symmetric under the [illegible] by

$$a=\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix}\,,\quad d=\frac{i}{\sqrt{3}}\begin{pmatrix}1&1&1\\1&\omega^2&\omega\\1&\omega&\omega^2\end{pmatrix}\,,\quad \omega\equiv\exp\left(\frac{2\pi i}{3}\right)\,,$$

and arbitrary permutations of doublets.

Σ(36) 3HDM

Consider a specific example: Σ(36)-symmetric 3HDM.

$$\begin{aligned} V &= -m^2\left(\phi_1^\dagger\phi_1+\phi_2^\dagger\phi_2+\phi_3^\dagger\phi_3\right)+\lambda_0\left(\phi_1^\dagger\phi_1+\phi_2^\dagger\phi_2+\phi_3^\dagger\phi_3\right)^2 \\ &\quad +\lambda_1\Big\{(\phi_1^\dagger\phi_1)^2+(\phi_2^\dagger\phi_2)^2+(\phi_3^\dagger\phi_3)^2-(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)-(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)-(\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1) \\ &\qquad\qquad +3(|\phi_1^\dagger\phi_2|^2+|\phi_2^\dagger\phi_3|^2+|\phi_3^\dagger\phi_1|^2)\Big\} \\ &\quad +\lambda_2\left(|\phi_1^\dagger\phi_2-\phi_2^\dagger\phi_3|^2 \dots |^2\right). \end{aligned}$$

Question

Q4.3: check it!

It is symmetric under the … by

$$a=\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix},\quad d=\frac{i}{\sqrt{3}}\begin{pmatrix}1&1&1\\1&\omega^2&\omega\\1&\omega&\omega^2\end{pmatrix},\quad \omega\equiv\exp\left(\frac{2\pi i}{3}\right),$$

and arbitrary permutations of doublets. What are its global minima?

$\Sigma(36)$ 3HDM

First simplify it as $V = -m^2 R + \lambda_0 R^2 + \lambda_1 X_1 + \lambda_2 X_2$, where

$$
\begin{aligned}
R &= \phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3\,,\\
X_1 &= (\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2 + (\phi_3^\dagger\phi_3)^2 - (\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) - (\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)\\
&\quad -(\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1) + 3(|\phi_1^\dagger\phi_2|^2 + |\phi_2^\dagger\phi_3|^2 + |\phi_3^\dagger\phi_1|^2)\,,\\
X_2 &= |\phi_1^\dagger\phi_2 - \phi_2^\dagger\phi_3|^2 + |\phi_2^\dagger\phi_3 - \phi_3^\dagger\phi_1|^2 + |\phi_3^\dagger\phi_1 - \phi_1^\dagger\phi_2|^2\,.
\end{aligned}
$$

At the neutral vacuum, $X_1 = R^2$ (**Q4.4: check it!**), so that the potential at neutral vacuum can be simplified further:

$$V = -m^2 R + R^2(\lambda_0 + \lambda_1 + \lambda_2 x_2)\,, \quad \text{where} \quad x_2 = \frac{X_2}{R^2}\,.$$

Σ(36) 3HDM

First simplify it as $V = -m^2 R + \lambda_0 R^2 + \lambda_1 X_1 + \lambda_2 X_2$, where

$$\begin{aligned} R &= \phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3\,, \\ X_1 &= (\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2 + (\phi_3^\dagger\phi_3)^2 - (\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) - (\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) \\ &\qquad -(\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1) + 3(|\phi_1^\dagger\phi_2|^2 + |\phi_2^\dagger\phi_3|^2 + |\phi_3^\dagger\phi_1|^2)\,, \\ X_2 &= |\phi_1^\dagger\phi_2 - \phi_2^\dagger\phi_3|^2 + |\phi_2^\dagger\phi_3 - \phi_3^\dagger\phi_1|^2 + |\phi_3^\dagger\phi_1 - \phi_1^\dagger\phi_2|^2\,. \end{aligned}$$

At the neutral vacuum, $X_1 = R^2$ (**Q4.4: check it!**), so that the potential at neutral vacuum can be simplified further:

$$V = -m^2 R + R^2(\lambda_0 + \lambda_1 + \lambda_2 x_2)\,, \quad \text{where} \quad x_2 = \frac{X_2}{R^2}\,.$$

The potential depends on two independent degrees of freedom: R and x_2.

$\Sigma(36)$ 3HDM

First solve for optimal R at any given x_2: $R = m^2/2(\lambda_0 + \lambda_1 + \lambda_2 x_2)$.

Σ(36) 3HDM

First solve for optimal R at any given x_2: $R = m^2/2(\lambda_0 + \lambda_1 + \lambda_2 x_2)$.
Then potential becomes a function of single variable x_2:

$$V_{\mathrm{opt.}R}(x_2) = -\frac{m^2}{4(\lambda_0 + \lambda_1 + \lambda_2 x_2)}\,.$$

So, the global minimum is attained at minimal (but positive) value of $\lambda_0 + \lambda_1 + \lambda_2 x_2$.

$\Sigma(36)$ 3HDM

First solve for optimal R at any given x_2: $R = m^2/2(\lambda_0 + \lambda_1 + \lambda_2 x_2)$.
Then potential becomes a function of single variable x_2:

$$V_{\mathrm{opt.}R}(x_2) = -\frac{m^2}{4(\lambda_0 + \lambda_1 + \lambda_2 x_2)}\,.$$

So, the global minimum is attained at minimal (but positive) value of $\lambda_0 + \lambda_1 + \lambda_2 x_2$.

Depending on sign of λ_2, we have two possibilities:

- $\lambda_2 > 0 \to$ the global minimum is when x_2 is smallest $\to$ $x_2 = 0$,
- $\lambda_2 < 0 \to$ the global minimum is when x_2 is largest $\to$ $x_2 = 1$.

$\Sigma(36)$ 3HDM

First solve for optimal R at any given x_2: $R = m^2/2(\lambda_0 + \lambda_1 + \lambda_2 x_2)$.
Then potential becomes a function of single variable x_2:

$$V_{\mathrm{opt}.R}(x_2) = -\frac{m^2}{4(\lambda_0 + \lambda_1 + \lambda_2 x_2)}\,.$$

So, the global minimum is attained at minimal (but positive) value of $\lambda_0 + \lambda_1 + \lambda_2 x_2$.

Depending on sign of λ_2, we have two possibilities:

- $\lambda_2 > 0 \to$ the global minimum is when x_2 is smallest $\to x_2 = 0$,
- $\lambda_2 < 0 \to$ the global minimum is when x_2 is largest $\to x_2 = 1$.

Question

Q4.5: check it!

$\Sigma(36)$ 3HDM

What remains is to find vev alignments corresponding to these two x_2 points.

- Up to an overall factor, the global minima for the $\Sigma(36)$ 3HDM with $\lambda_2 > 0$ are

$$(1,\,0,\,0)\,,\quad (0,\,1,\,0)\,,\quad (0,\,0,\,1)\,.$$
$$(1,\,1,\,1)\,,\quad (1,\,\omega,\,\omega^2)\,,\quad (1,\,\omega^2,\,\omega)\,.$$

- Up to an overall factor, the global minima for the $\Sigma(36)$ 3HDM with $\lambda_2 < 0$ are

$$(\omega,\,1,\,1)\,,\quad (1,\,\omega,\,1)\,,\quad (1,\,1,\,\omega)\,,$$
$$(\omega^2,\,1,\,1)\,,\quad (1,\,\omega^2,\,1)\,,\quad (1,\,1,\,\omega^2)\,.$$

We solved the minimization problem.

$\Sigma(36)$ 3HDM

What remains is to find vev alignments corresponding to these two x_2 points.

- Up to an overall factor, the global minima for the $\Sigma(36)$ 3HDM with $\lambda_2 > 0$ are

$$(1,\,0,\,0)\,,\quad (0,\,1,\,0)\,,\quad (0,\,0,\,1)\,.$$
$$(1,\,1,\,1)\,,\quad (1,\,\omega,\,\omega^2)\,,\quad (1,\,\omega^2,\,\omega)\,.$$

Question

Q4.6: derive these vev alignments.

- Up to an ove … 36) 3HDM with $\lambda_2 < 0$ are

$$(\omega,\,1,\,1)\,,\quad (1,\,\omega,\,1)\,,\quad (1,\,1,\,\omega)\,,$$
$$(\omega^2,\,1,\,1)\,,\quad (1,\,\omega^2,\,1)\,,\quad (1,\,1,\,\omega^2)\,.$$

We solved the minimization problem.

Symmetry breaking and CKM matrix

Scalar symmetries and their breaking patterns in NHDM can have important effects on fermion sector. Quark and neutrino phenomenology has received a lot of attention within NHDM, but vast majority of works deal with specific models and their testing against data.

Here, I would like to show an example of general results which can be derived in NHDM *[Gonzalez Felipe, Ivanov, Nishi, Serodio, Silva, 2014]*.

Symmetry breaking and CKM matrix

Scalar symmetries and their breaking patterns in NHDM can have important effects on fermion sector. Quark and neutrino phenomenology has received a lot of attention within NHDM, but vast majority of works deal with specific models and their testing against data.

Here, I would like to show an example of general results which can be derived in NHDM *[Gonzalez Felipe, Ivanov, Nishi, Serodio, Silva, 2014]*.

No-Go theorem for quark sector in NHDM

Given a group G acting in the quark flavour space, the only way to avoid unrealistic CKM mixing matrix is that vevs $\langle\phi_i\rangle$ break completely the group G.

"Unrealistic CKM" $=$ block-diagonal, with no CP-violation and one mixing angle.

Symmetry breaking and CKM matrix

- Suppose the NHDM with quarks is invariant under a flavour group G, which acts in four spaces: Q_L, d_R, u_R, and ϕ_i. Within each space, it acts by certain representation, not necessarily irreducibe.

Symmetry breaking and CKM matrix

- Suppose the NHDM with quarks is invariant under a flavour group G, which acts in four spaces: Q_L, d_R, u_R, and ϕ_i. Within each space, it acts by certain representation, not necessarily irreducibe.
- Suppose that after EWSB, the vacuum remains invariant under a residual symmetry, for example, under $g \in G$. Then, the quark mass matrices also remain symmetric:

$$G_L^\dagger M_d G_{dR} = M_d \,, \quad G_L^\dagger M_u G_{uR} = M_u \,,$$

where G_L, G_{dR}, and G_{uR} are transformation matrices of the element g in the spaces Q_L, d_R, u_R, respectively.

Symmetry breaking and CKM matrix

- Suppose the NHDM with quarks is invariant under a flavour group G, which acts in four spaces: Q_L, d_R, u_R, and ϕ_i. Within each space, it acts by certain representation, not necessarily irreducibe.
- Suppose that after EWSB, the vacuum remains invariant under a residual symmetry, for example, under $g \in G$. Then, the quark mass matrices also remain symmetric:

$$G_L^\dagger M_d G_{dR} = M_d \,, \quad G_L^\dagger M_u G_{uR} = M_u \,,$$

  where G_L, G_{dR}, and G_{uR} are transformation matrices of the element g in the spaces Q_L, d_R, u_R, respectively.
- If $G_L \not\propto \mathbb{I}_3$, then matrices M_d and M_u will have a common eigenvector, and the CKM matrix is block-diagonal.
- The only possibility compatible with good CKM is that $G_L \propto \mathbb{I}_3 \to G_\phi = \mathbb{I}$ $\to$ no symmetry remains after EWSB.

Possible types of CP-violation in NHDM

Reminder:

CP conservation means invariance under any generalized CP (gCP) transformation = CP accompanied by a unitary transformation.

Possible types of *CP*-violation in NHDM

Reminder:

CP conservation means invariance under any generalized *CP* (*gCP*) transformation = *CP* accompanied by a unitary transformation.

1. If the scalar potential and the vacuum are *CP*-symmetry, then we have *CP*-conserving Higgs sector. In this case, *CP*-violation arises from complex Yukawa couplings, like in the SM.

Possible types of *CP*-violation in NHDM

Reminder:

CP conservation means invariance under any generalized *CP* (*gCP*) transformation = *CP* accompanied by a unitary transformation.

1. If the scalar potential and the vacuum are *CP*-symmetry, then we have *CP*-conserving Higgs sector. In this case, *CP*-violation arises from complex Yukawa couplings, like in the SM.

2. *CP* symmetry is explicitly broken in the scalar potential, when the potential is not symmetric under any *gCP* transformation.

Whether vevs have relative phases or not, is irrelevant because physical Higgses anyway do not possess definite *CP*.

Possible types of *CP*-violation in NHDM

3. *CP* symmetry is present in the scalar potential, but it is spontaneously broken at the vacuum. This happens when in certain basis the potential has all real coefficients but vevs acquire relative phases, and when there is no *gCP* symmetry left in the vacuum which kills these phases.

Possible types of CP-violation in NHDM

3. CP symmetry is present in the scalar potential, but it is spontaneously broken at the vacuum. This happens when in certain basis the potential has all real coefficients but vevs acquire relative phases, and when there is no gCP symmetry left in the vacuum which kills these phases.

Example: in the $\Sigma(36)$ 3HDM, the potential is invariant under the usual CP symmetry, but some of the vev alignments are not. But there remains a gCP transformation which leaves the vacuum invariant. For example, $(\omega, 1, 1)$ is invariant under CP followed by transformation d.

Thus, the model is not spontaneously CP violating.

Possible types of CP-violation in NHDM

3. CP symmetry is present in the scalar potential, but it is spontaneously broken at the vacuum. This happens when in certain basis the potential has all real coefficients but vevs acquire relative phases, and when there is no gCP symmetry left in the vacuum which kills these phases.

Example: in the $\Sigma(36)$ 3HDM, the potential is invariant under the usual CP symmetry, but some of the vev alignments are not. But there remains a gCP transformation which leaves the vacuum invariant. For example, $(\omega, 1, 1)$ is invariant under CP followed by transformation d.

Thus, the model is not spontaneously CP violating.

Question

Q4.7: check it!

Possible types of *CP*-violation in NHDM

4. If in the previous case we have spontaneous *CP*-violation with fixed phases, then we deal with geometrical *CP*-violation *[Branco, Gerard, Grimus, 1984]*. This possibility is attractive because the phases (and therefore *CP*-violating effects) are not sensitive to changes of free parameters of the potential.

Possible types of CP-violation in NHDM

4. If in the previous case we have spontaneous CP-violation with fixed phases, then we deal with geometrical CP-violation *[Branco, Gerard, Grimus, 1984]*. This possibility is attractive because the phases (and therefore CP-violating effects) are not sensitive to changes of free parameters of the potential.

The feature was absent in 2HDM; it appears starting from 3HDM.

Example is again the vev alignment $(\omega, 1, 1)$ but for a less symmetric model $\Delta(54)$ 3HDM. In that case, generator d is absent, so that $(\omega, 1, 1)$ does not have gCP symmetry anymore.

Lecture 4 Summary

- There exists, at least in principle, a method to list all possible non-abelian symmetry groups in any bSM model: we use the abelian group results and apply some advanced pure group theory. Unfortunately, this method is rather difficult. The scalar sector of 3HDM is the only example so far where it was successfully applied and led to non-trivial results.

Lecture 4 Summary

- There exists, at least in principle, a method to list all possible non-abelian symmetry groups in any bSM model: we use the abelian group results and apply some advanced pure group theory. Unfortunately, this method is rather difficult. The scalar sector of 3HDM is the only example so far where it was successfully applied and led to non-trivial results.
- Breaking of a large symmetry group in scalar sector of NHDM cannot be arbitrary. It follows certain patterns and satisfies some relations, with some open questions remaining. This breaking patterns have important consequences for fermions and for *CP*-violation..

Overall summary

Until recently, the study of various phenomena offered by rich structure of 2HDM and NHDM was restricted to the domain of simplified models, isolated examples, numerical methods, haphazard ideas, or pure guess.

> Now, little by little, there emerges a whole set of tools which make it possible to study NHDM in a systematic way.

Of course, the big question is how these methods and their results meet the real world experimental data. But at least we have a chance to study this question systematically.