Marcinkiewicz multipliers in products of Heisenberg groups

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Abstract

Let $G = G_1 \times \cdots \times G_n$ be a product of $H$-type groups, and let, for each $j = 1, \ldots, n$,

$$X_1^j, Y_1^j, \ldots, X_n^j, Y_n^j, Z_1^j, \ldots, Z_d^j$$

be a basis for the Lie algebra $\mathfrak{g}_j$ of $G_j$ such that $Z_1^j, \ldots, Z_d^j$ is a basis for its center $\mathfrak{z}_j$.

Given a bounded function $m$ on $\mathbb{R}_+^n \times \mathbb{R}^d$, where $d = d_1 + \cdots + d_n$, we can define the Marcinkiewicz operator $\mathcal{M} = m(L_1, \ldots, L_n, -iZ_1^1, \ldots, -iZ_n^d)$ through the spectral theorem, and such operator is bounded on $L^2(G)$. Here $L_j$ is the partial sublaplacian

$$L_j = \sum_{k=1}^{n_j} -((X_k^j)^2 + (Y_k^j)^2)$$

on $G_j$. We study the boundedness of $\mathcal{M}$ on $L^p(G_K)$, given sufficient regularity conditions on the function $m$.

In particular, we study the case where each $G_j = \mathbb{H}_1$, and we use the result in such product to study Marcinkiewicz operators on quotients $(\mathbb{H}_1)^n/K$, where $K$ is a central subgroup of $(\mathbb{H}_1)^n$. 

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Introduction

The purpose of this thesis is to study the boundedness of Marcinkiewicz operators on a special class of step-2 nilpotent groups, in particular products of $H$-type groups and groups obtained from taking the quotient of a product of $(2 + 1)$-dimensional Heisenberg groups with respect to a central subgroup.

The origin of this study is the following fact: If $T$ is a translation invariant operator acting, say, on Schwartz functions on $\mathbb{R}^n$, then $T$ is given by convolution with some distribution $K$, i.e.

$$T f = f \ast K.$$ 

Moreover, if $T$ is a bounded operator on $L^2(\mathbb{R}^n)$, then the the Fourier transform $m = \hat{K}$ of the distribution $K$ is a bounded function on $\mathbb{R}^n$ (see Chapter 1 in [Ste70], Theorem 3.18). A natural question to ask is whether the operator $T$ is bounded on $L^p(\mathbb{R}^n)$ for $p \neq 2$. If $T$ is bounded on $L^p$ then $m$ is called a multiplier of type $p$-$p$. It is well known that if the function $m$ is sufficiently regular, say, $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and, for any multi-index $\alpha$,

$$|\partial^\alpha m(\xi)| \leq A_\alpha |\xi|^{-|\alpha|},$$

(0.1)

then $T$ is bounded on $L^p(\mathbb{R}^n)$, for any $1 < p < \infty$. Results of this type are originally due to Marcinkiewicz [Mar39] in the periodic setting, and to Mihlin in the nonperiodic setting [Mih56].
One then asks for the optimal regularity conditions that \( m \) must satisfy for \( T \) to be bounded. For example, one can ask for the smallest integer \( l \) such that, if (0.1) is satisfied for \( |\alpha| \leq l \), then \( T \) is bounded on \( L^p(\mathbb{R}^n) \). In fact, if \( l > n/2 \) and (0.1) holds for \( |\alpha| \leq l \), then the distribution \( K \) agrees with a function \( K(x) \) away from the origin, it is locally integrable and

\[
\int_{|x| \geq 2|y|} |K(x - y) - K(x)| dx \leq A
\]

(0.2)

for all \( y \neq 0 \) and some constant \( A \) which depends only on the constants \( A_\alpha \) in (0.1), which in turn implies that \( T \) is of weak type (1,1). See, for instance, Section 4.4 of Chapter VI in [Ste93]. It is possible to give weaker conditions that make \( m \) a multiplier of type \( p-p \). The following theorem, for example, is due to Hörmander ([Hör60]).

**Theorem** (Hörmander). Let \( l \) be the smallest integer greater than \( n/2 \). If, for every multi-index \( \alpha \) with \( |\alpha| \leq l \),

\[
\sup_{r>0} r^{2|\alpha|-n} \int_{r<|\xi|<2r} |\partial_\xi^\alpha m(\xi)|^2 d\xi < \infty,
\]

then \( T \) is bounded on \( L^p(\mathbb{R}^n) \), \( 1 < p < \infty \).

A slightly weaker condition is given by the following: Fix \( \eta \in C_0^\infty(\mathbb{R}) \) supported in \((1/2, 2)\). Then, if there exists \( s > n/2 \) such that

\[
\sup_{r>0} ||\eta(|\xi|)m(r\xi)||_{L^2_s(\mathbb{R}^n)} < \infty,
\]

(0.3)

then \( m \) is a multiplier of type \( p-p \). Here, \( L^2_s(\mathbb{R}^n) \) denotes the Sobolev space of order \( s \), with norm

\[
||f||_{L^2_s(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}.
\]

The proofs of these results follow from the Plancherel theory of the Fourier transform.
as well as its homogeneity, i.e. the fact that, if \( g(x) = f(rx) \) for \( r > 0 \), then

\[
\hat{g}(\xi) = r^{-n} \hat{f}(\xi/r).
\]

See Section 4.4 of Chapter VI in [Ste93] for the details. Note that, in particular, if \( m \) is radial, then \( m(\xi) = m_0(|\xi|) \), and hence the condition

\[
\sup_{r>0} \|\eta(|\xi|)m_0(r\xi)\|_{L^2_\mathbb{R}} < \infty
\]

is sufficient for the boundedness of \( T \) on \( L^p(\mathbb{R}^n) \). Moreover, by the spectral theorem (see, for instance, [Rud73]), \( T \) is precisely the operator \( m_0(\sqrt{-\Delta}) \), where \( \Delta \) is the Laplacian on \( \mathbb{R}^n \).

We can thus consider a more general setting. Let \( \mathfrak{g} \) be a nilpotent Lie algebra. We say that \( \mathfrak{g} \) is \textit{stratified} if we can write

\[
\mathfrak{g} = \bigoplus_j V_j,
\]

as a vector space, such that \( V_1 \) generates \( \mathfrak{g} \) and \( [V_i, V_j] \subset V_{i+j} \) for each \( i, j \). We then say that the Lie group \( G \) is stratified if its Lie algebra is stratified. We can define a family of \textit{dilations} \( \delta_t, t > 0 \), on \( G \) by

\[
\delta_t(\exp \sum_j X_j) = \exp(\sum_j t^j X_j),
\]

where each \( X_j \in V_j \) and \( \exp : \mathfrak{g} \to G \) is the exponential map (which is a diffeomorphism; see [Kna96]). The family \( \{\delta_t\} \) forms a semigroup of automorphisms on \( G \), and a group provided with such a family of dilations is called \textit{homogeneous}. 

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The **homogeneous dimension** of $G$ is given by

$$ Q = \sum_j j \dim V_j. $$

Note that $Q \geq D$, where $D$ is the real dimension of $G$ (as a manifold), with equality only when $g$ is abelian.

The reader may find a wide exposition of stratified groups, and more general homogeneous groups, in [FS82]. Examples of stratified groups are given by $\mathbb{R}^n$, with abelian Lie algebra and dilations equal to scalar multiplication, and the **Heisenberg group** $\mathbb{H}_n$, which is the set $\mathbb{C}^n \times \mathbb{R}$ with product

$$(z, t) \cdot (w, s) = (z + w, t + s + \frac{1}{2} \Im(z \cdot \bar{w})),$$

where $z \cdot \bar{w} = z_1 \bar{w}_1 + \ldots + z_n \bar{w}_n$ and $\Im z$ denotes the imaginary part of $z$. The Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$ has a basis $X_1, Y_1, \ldots, X_n, Y_n, T$ with

$$ [X_j, Y_k] = -\delta_{jk} T, $$

all other equal to zero. Thus, in this case, $D = 2n + 1$ and $Q = 2n + 2$. The dilations are explicitly given by

$$ \delta_r(z, t) = (rz, r^2 t). $$

Chapter 1 of this thesis has more details on $\mathbb{H}_n$.

Let $l = \dim V_1$ and $X_1, \ldots, X_l$ a basis for $V_1$. Then, the **sublaplacian** on $G$ is given by

$$ \mathcal{L} = -\sum_{j=1}^l X_j^2. $$

This operator is formally self-adjoint, nonnegative, homogeneous of order 2 with respect to the family $\{\delta_t\}$ and, by a theorem of Hörmander [Hör67], hypoelliptic.
It is also densely defined in $L^2(G)$, and, by the spectral theorem, given a bounded function $m$ on $\mathbb{R}_+ = (0, \infty)$ we can define the operator $m(\mathcal{L})$, which is bounded on $L^2(G)$. We thus ask whether $m(\mathcal{L})$ is bounded on $L^p(G)$ for $p \neq 2$; in such case we say that $m$ is an \textit{spectral multiplier} of type $p$-$p$.

As above, if $m$ satisfy certain regularity conditions, then it is an spectral multiplier of type $p$-$p$, for $1 < p < \infty$. In particular, one has the following theorem:

**Theorem** (Christ, De-Michelle, Mauceri). \textit{If for some $s > Q/2$}

$$\sup_{r > 0} ||\eta(\xi)m(r\xi)||_{L^2_s(\mathbb{R})} < \infty,$$

then $m$ is an \textit{spectral multiplier} of type $p$-$p$, for $1 < p < \infty$.

This theorem was proved by De-Michelle and Mauceri in [DMM87], and, independently, by Christ in [Chr91]. The proof involves weighted $L^2$ estimates of the convolution kernel of $m(\mathcal{L})$, as well as the homogeneity of the operator $\mathcal{L}$. The $L^2$ estimates needed are obtained through estimates of the \textit{heat kernel} on $G$. See [DMM87], and the references therein, for the details.

This theorem implies that regularity conditions of order $Q/2$, where $Q$ is the homogeneous dimension of the group $G$, are sufficient for $m$ to be a multiplier. However, Müller and Stein [MS94] proved a stronger result: regularity of order $D/2$, where $D$ is the real dimension, is enough in the case of the Heisenberg group.

**Theorem** (Müller, Stein). \textit{If for some $s > D/2 = (2n + 1)/2$}

$$\sup_{r > 0} ||\eta(\xi)m(r\xi)||_{L^2_s(\mathbb{R})} < \infty,$$

then the operator $m(\mathcal{L})$ is bounded on $L^p(\mathbb{H}_n)$, for $1 < p < \infty$.

They also proved that this theorem is sharp, in the sense that regularity of order $D/2$ is also a necessary condition in such generality. The proof now uses sharper $L^2$
estimates, which are obtained by using the Fourier analysis of $\mathbb{H}_n$. See [MS94] for the details. The Fourier analysis of this group will be reviewed briefly in the next chapter of this thesis.

Hebisch [Heb93] extended Müller and Stein’s result to $H$-type groups. A step-2 connected and simply connected nilpotent group $G$ is called an $H$-type group (or generalized Heisenberg group) if its Lie algebra $\mathfrak{g}$ (itself called an $H$-type algebra) satisfies the following condition: assume $\mathfrak{g}$ is provided with an inner product, let $\mathfrak{z} = [\mathfrak{g}, \mathfrak{g}]$ and let $\mathfrak{w}$ be the orthogonal complement to $\mathfrak{z}$. Then, given $X \in \mathfrak{w}$ with $\|X\| = 1$, the mapping $\text{ad}_X^* : \mathfrak{z}^* \mapsto \mathfrak{w}^*$ is an isometry.

Theorem (Hebisch). Let $G$ be an $H$-type group of real dimension $D$. If for some $s > D/2$

$$\sup_{r > 0} \|\eta(\xi) m(r\xi)\|_{L^p_s(\mathbb{R})} < \infty,$$

then the operator $m(L)$ is bounded on $L^p(G)$, for $1 < p < \infty$.

Its proof involves heat kernel estimates, as well as the use of square functions. See [Heb93], and [HZ95], which provides a simplified proof, for the details.

Müller, Ricci and Stein [MRS95, MRS96] (see also [Ste95]) considered more general Marcinkiewicz operators on $H$-type groups $G$. If $T_1, \ldots, T_d$ form a basis for the center $\mathfrak{z}$ of the Lie algebra of the $(2n + d)$-dimensional group $G$, they studied the operator

$$m(L, -iT_1, \ldots, -iT_d),$$

and proved in [MRS96] that a regularity condition of order $D/2$ satisfied by $m$ is also sufficient for the boundedness of this operator. The condition is now stronger than in the previous results, since it now involves multiparameter Sobolev estimates. Precisely, they proved the following theorem.

Theorem (Müller, Ricci, Stein). Define the multiparameter Sobolev norm, for a func-
tion $f$ on $\mathbb{R} \times \mathbb{R}^n$ and $s, s' > 0$,

$$||f||_{L^2_{s,s'}} = \left( \int_{\mathbb{R} \times \mathbb{R}^n} (1 + |2\pi \xi|^2)^s \prod_{j=1}^d (1 + |2\pi \xi_j|^2 + |2\pi \zeta_j|^2)^{s'} |\hat{f}(\xi, \zeta)|^2 d\xi d\zeta \right)^{1/2},$$

where $\hat{f}$ is the Fourier transform of $f$. Let $m$ be a bounded function on $\mathbb{R}_+ \times \mathbb{R}^d$. If for some $s > n$ and $s' > 1/2$

$$\sup_{r_j > 0, j=0,1,...,d} ||\eta(\xi)\eta(\zeta_1) \cdots \eta(\zeta_d)m(r_0, r_1, \ldots, r_d, \zeta_0)||_{L^2_{s,s'}} < \infty,$$

then the operator $m(L, -iT_1, \ldots, -iT_d)$ is bounded on $L^p(G)$ for $1 < p < \infty$.

Note that, since $D = 2n + d$ and the definition of the norm $|| \cdot ||_{L^2_{s,s'}}$ involves $d$ factors of order $s'$, then the regularity condition required for the boundedness of the operator in the theorem is of order $n + d/2 = D/2$. The proof of this theorem makes use of Littlewood-Paley theory on the group $G$ besides its Fourier analysis. See [MRS96] for the details. We will prove in this work a product version of this theorem for the case of the Heisenberg group, whose proof is similar to the proof of Müller-Ricci-Stein’s theorem.

Veneruso [Ven00b] considered, on the $(2n + 1)$-dimensional Heisenberg group $\mathbb{H}_n$, the more general operator $m(L_1, \ldots, L_n, -iT)$ (see also [Ven00a], [Fra01b], [Fra01a]). His result also requires a regularity condition stated in terms of a multiparameter Sobolev estimate. It can be written in the following form.

**Theorem** (Veneruso). Define, for a function $f$ on $\mathbb{R}^n \times \mathbb{R}$ and $s, s > 0$,

$$||f||_{L^2_{s,s'}} = \left( \int_{\mathbb{R}^n \times \mathbb{R}} (1 + |2\pi \xi_1|^2)^s \cdots (1 + |2\pi \xi_n|^2)^s (1 + |2\pi \xi|^2 + |2\pi \zeta_j|^2)^{s'} |\hat{f}(\xi, \zeta)|^2 d\xi d\zeta \right)^{1/2}.$$
Then, if for some $s > 1$ and $s' > 1/2$,

$$
\sup_{r_j > 0} \left\| \eta(\xi_1) \cdots \eta(\xi_n) \eta(|\xi|) m(r_1 \xi_1, \ldots, r_n \xi_n, r_{n+1} \xi) \right\|_{L^2_{s,s'}} < \infty,
$$

then the operator $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT)$ is bounded on $L^p(\mathbb{H}_n)$ for $1 < p < \infty$.

We observe that a multiparameter regularity condition of order $D/2$ in $m$ is sufficient for the boundedness of the operator $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT)$.

In [MS94] is also proved that, in the case of a product of Heisenberg groups $G = \mathbb{H}_{n_1} \times \cdots \times \mathbb{H}_{n_k}$, a regularity condition of order $D/2$, where $D$ is the real dimension of $G$, is sufficient for the boundedness of the operator $m(\mathcal{L})$, where $\mathcal{L}$ is sublaplacian on $G$. Hebisch and Zienkiewicz [HZ95] also extended Hebisch’ theorem on $H$-type groups to products of $H$-type groups, also proving that a regularity condition of order $D/2$ in $m$ is sufficient for the boundedness of $m(\mathcal{L})$.

In this thesis we extend these studies to Marcinkiewicz operators of the form

$$
T = m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT_1, \ldots, iT_n)
$$

on products of $H$-type groups $G = G_1 \times \cdots \times G_n$, where each operator $\mathcal{L}_j$ is the partial sublaplacian on each $G_j$, and $T_j = (Z_j^1, \ldots, Z_j^d_j)$ is the differential operator formed with a basis of the center of each Lie algebra $\mathfrak{g}_j$ of $G_j$. We also prove that a multiparameter regularity condition on $m$ of order $D/2$, where $D$ is the real dimension of $G$, is sufficient for the boundedness of the operator $T$.

In particular, we use explicitly the case of $G = (\mathbb{H}_1)^n$ to study Marcinkiewicz operator on central quotients $G/K$ of $G$, obtained by taking the quotient of $G$ with a proper $(n - d)$-dimensional central subgroup $K$ (see Chapter 3 for details). We do this by a “lift” of the operator $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT)$ on $G/K$ to a Marcinkiewicz operator $G$, with multiplier satisfying a regularity condition of order $3n/2$. 

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The work is organized as follows: In Chapter 1, we review the Fourier analysis of the Heisenberg group $\mathbb{H}_n$ and the product $(\mathbb{H}_1)^n$, and develop a Littlewood-Paley theory in such product. In Chapter 2, we prove a Marcinkiewicz theorem on the group $(\mathbb{H}_1)^n$. Although we later prove these results in more generality, we first present the Heisenberg group case as the proofs are simpler, besides the fact that they imply the results in Chapter 3.

In Chapter 3, we study the quotients $(\mathbb{H}_1)^n/K$, for central subgroups $K$. We develop the Fourier analysis on $(\mathbb{H}_1)^n/K$, and state the transference results needed to obtain Marcinkiewicz theorems on such groups. In Chapter 4 we develop the Fourier analysis on products of $H$-type groups, including Littlewood-Paley theory, analogous to the one developed in [MRS96]. Finally, in Chapter 5, we prove a product version of the main result of [MRS96].
Chapter 1

Fourier Analysis on products of Heisenberg groups

1.1 Products of Heisenberg groups

The Heisenberg group $\mathbb{H}_n$ is given by the set $\mathbb{C}^n \times \mathbb{R}$ with operation

$$(z, t) \cdot (w, s) = (z + w, t + s + \frac{1}{2} \Im(z \cdot \bar{w})), \quad z, w \in \mathbb{C}^n, \quad t, s \in \mathbb{R},$$

where $z \cdot \bar{w} = z_1 \bar{w}_1 + \ldots + z_n \bar{w}_n$ and $\Im(z)$ is the imaginary part of $z$. It is denoted by $\mathbb{H}_n$, is $(2n + 1)$-dimensional and its Lie algebra $\mathfrak{h}_n$ has basis $X_1, Y_1, \ldots, X_n, Y_n, T$ with

\[
X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, n
\]
\[
Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, n
\]
\[
T = \frac{\partial}{\partial t},
\]

and commutation relations

$$[X_j, Y_k] = -\delta_{jk} T,$$
all other equal to zero.

\( \mathbb{H}_n \) is a homogeneous group with dilations

\[
\delta_r(z, t) = (rz, r^2t), \quad r > 0,
\]

which are automorphisms of the group (it is usually written as \( r(z, t) \)), and homogeneous norm

\[
|(z, t)| = (|z|^4 + |t|^2)^{1/4},
\]

which satisfies \(|r(z, t)| = r|(z, t)|\), for \( r > 0 \). The homogeneous dimension of \( \mathbb{H}_n \) is given by \( Q = 2n + 2 \).

Note that, if \( dzdt \) is the Lebesgue measure on \( \mathbb{C}^n \times \mathbb{R} \), then \( dzdt \) is then the Haar measure on \( \mathbb{H}_n \). Also, for \( f \in L^1(\mathbb{H}_n) \),

\[
\int_{\mathbb{H}_n} f \circ \delta_r dzdt = \frac{1}{r^Q} \int_{\mathbb{H}_n} f dzdt.
\]

We can classify the irreducible unitary representations of \( \mathbb{H}_n \) (this is the Stone-von Neumann theorem, see [Fol89, Tay86, Tha98]). These are given by the 1-dimensional representations

\[
\pi_{(q,p)}(x, y, t) = e^{i(q \cdot x + p \cdot y)}, \quad q, p \in \mathbb{R}^n,
\]

which are trivial in the center of \( \mathbb{H}_n \), \( \{(0, 0, t) : t \in \mathbb{R}\} \), (we are viewing \( z \) as \( (x, y), x, y \in \mathbb{R}^n \)), and the so-called Schrödinger representations on \( L^2(\mathbb{R}^n) \) given by

\[
\pi_\lambda(x, y, t) \phi(\xi) = e^{i\lambda(t+x \cdot \xi + \frac{1}{2}x \cdot y)} \phi(\xi + y), \quad \lambda \in \mathbb{R}_+ = \mathbb{R} \setminus \{0\}, \quad \phi \in L^2(\mathbb{R}^n).
\]
For each function $f \in L^1(\mathbb{H}_n)$ we define its Fourier transform $\hat{f}$ as the operator
\[
\hat{f}(\lambda) = \int_{\mathbb{H}_n} f(z,t) \pi_\lambda(z,t) dz dt, \quad \lambda \in \mathbb{R}_+.
\] (1.3)

We have the following.

**Theorem 1.1** (Plancherel theorem for $\mathbb{H}_n$). Let $f \in L^1 \cap L^2(\mathbb{H}_n)$. Then
\[
||f||_{L^2(\mathbb{H}_n)} = (2\pi)^{-\frac{n+1}{2}} \int_{\mathbb{R}_+} \text{tr} (\hat{f}(\lambda)^* \hat{f}(\lambda)) |\lambda|^n d\lambda.
\] (1.4)

Here $\hat{f}(\lambda)^*$ is the adjoint of $\hat{f}(\lambda)$ and $\text{tr}(T)$ denotes the trace of the operator $T$. The measure
\[
(2\pi)^{-\frac{n+1}{2}} |\lambda|^n d\lambda
\]
is called the **Plancherel measure** on $\mathbb{H}_n$. A proof of this theorem can be found in either of the references [Fol89, Tay86, Tha98]. By polarizing (1.4) we obtain the following inversion formula.

**Corollary 1.2** (Inversion formula for $\mathbb{H}_n$). For suitable functions$^2$ $f$ on $\mathbb{H}_n$,
\[
f(z,t) = (2\pi)^{-\frac{n+1}{2}} \int_{\mathbb{R}_+} \text{tr} (\pi_\lambda(z,t)^* \hat{f}(\lambda)) |\lambda|^n d\lambda.
\] (1.5)

The sublaplacian on $\mathbb{H}_n$ is given by
\[
\mathcal{L} = -\sum_{j=1}^n (X_j^2 + Y_j^2).
\]

$^1$We ignore the representations $\pi_{(q,p)}$

$^2$Say, in the Schwartz space $\mathcal{S}(\mathbb{H}_n)$, defined as the set of smooth functions $f$ on $\mathbb{H}_n$ such that
\[
\sup_{(z,t) \in \mathbb{H}_n} |(z,t)^{p_1} X_1^{q_1} \cdots X_n^{q_n} T^r f(z,t)| < \infty
\]
for each $p, q_1, \ldots, q_n, r \in \mathbb{N}$. 

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Since the representations $\pi_\lambda$ map $X_j, Y_j$ to

$$
\pi_\lambda(X_j) = i\lambda_j, \quad \pi_\lambda(Y_j) = \frac{\partial}{\partial \xi_j},
$$

(acting on functions on $\mathbb{R}^n$) we obtain that

$$
\pi_\lambda(L) = -\Delta + \lambda^2|\xi|^2 = H_\lambda,
$$

(1.6)

where $\Delta$ is the Laplacian in $\mathbb{R}^n$. The operator (1.6) is called the scaled harmonic oscillator. The eigenfunctions of $H_\lambda$ are given by

$$
\Phi_{\alpha,\lambda}(\xi) = |\lambda|^{\frac{n}{2}} \prod_{j=1}^{n} h_{\alpha_j}(|\lambda|^{\frac{1}{2}} \xi_j),
$$

where $\alpha \in \mathbb{N}^n$ and $h_k(t)$ is the normalized Hermite function defined by

$$
h_k(t) = (2^k \sqrt{\pi} k!)^{-\frac{1}{2}} H_k(t) e^{-\frac{1}{2} t^2},
$$

where $H_k$ is the Hermite polynomial of order $k$,

$$
H_k(t) = (-1)^k e^{t^2} \frac{d^k}{dt^k} e^{-t^2}.
$$

See [Tha93] for a thorough study of Hermite functions. The eigenvalues of $H_\lambda$ are then given by $|\lambda|(2|\alpha| + n)$, $\lambda \in \mathbb{R}_+$, $\alpha \in \mathbb{N}$, where

$$
|\alpha| = \alpha_1 + \ldots + \alpha_n.
$$

A system of eigenfunctions for $L$ on $\mathbb{H}_n$ is then given by

$$
\phi_{\alpha,\lambda}(z, t) = (2\pi)^{-\frac{n}{2}} e^{-i\lambda \cdot z} \prod_{j=1}^{n} L_{\alpha_j} \left( \frac{1}{2} |\lambda||z_j|^2 \right) e^{-\frac{1}{2} |\lambda||z_j|^2},
$$

(1.7)
where \( L_k(t) \) is the Laguerre polynomial

\[
L_k(t) = \frac{1}{k!} e^t \frac{d^k}{dt^k} (e^{-t} t^k),
\]

(See [Str91, Tha93, Tha98].) From the Fourier inversion formula we obtain the following orthogonal expansion in \( L^2(\mathbb{H}_n) \).

**Theorem 1.3.** For \( f \in L^2(\mathbb{H}_n) \) we have the expansion

\[
f(z, t) = c_n \sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}^*} f \ast \phi_{\alpha, \lambda}(z, t)|\lambda|^n d\lambda,
\]

where the constant \( c_n \) depends only on \( n \).

See [Str91] or [Tha98] for a proof of Theorem 1.3.

Consider now the group \( G = (\mathbb{H}_1)^n \), i. e. the product of \( n \) \((2 + 1)\)-dimensional Heisenberg groups. We will denote it as the set \( \mathbb{C}^n \times \mathbb{R}^n \), with operation

\[
(z, t) \cdot (w, s) = (z + w, t + s + \frac{1}{2} \Im(z \cdot \bar{w})), \quad z, w \in \mathbb{C}^n, \quad t, s \in \mathbb{R}^n,
\]

where \( z \cdot \bar{w} = (z_1 \bar{w}_1, \ldots, z_n \bar{w}_n) \). Its 3\( n \)-dimensional Lie algebra has then basis \( X_1, Y_1, \ldots, X_n, Y_n, T_1, \ldots, T_n \), where

\[
X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t_j}, \quad j = 1, \ldots, n
\]

\[
Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t_j}, \quad j = 1, \ldots, n
\]

\[
T_j = \frac{\partial}{\partial t_j}, \quad j = 1, \ldots, n
\]

and the only nonzero commutators are

\[
[X_j, Y_j] = -T_j, \quad j = 1, \ldots, n.
\]
The unitary Schrödinger representations of \( G \) on \( L^2(\mathbb{R}^n) \) are given by the operators \( \pi^\lambda_n \), for \( \lambda \in \mathbb{R}^n \), where \( \pi^\lambda_n(z, t) \) is the tensor product

\[
\pi^\lambda_n(z_1, t_1) \otimes \cdots \otimes \pi^\lambda_n(z_n, t_n),
\]

i.e., for \( \phi \in L^2(\mathbb{R}^n) \), \( \xi \in \mathbb{R}^n \), and \( (z, t) = (x, y, t) \in \mathbb{C}^n \times \mathbb{R}^n \),

\[
\pi^\lambda_n(x, y, t)\phi(\xi) = \exp \left\{ i \left( \lambda \cdot t + \sum_{j=1}^n (x_j\xi_j + \frac{1}{2}x_jy_j\lambda_j) \right) \right\} \phi(\xi + y).
\]

The group Fourier transform is then, for \( f \in L^1(G) \), the operator \( \hat{f}(\lambda) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \) given by

\[
\hat{f}(\lambda)(\phi) = \int_G f(z, t)\pi^\lambda_n(z, t)\phi dz dt.
\]

Theorem 1.1 and Corollary 1.2 imply then the following result on \( G \).

**Theorem 1.4.** 1. Let \( f \in L^1 \cap L^2(G) \). Then

\[
\|f\|_{L^2((\mathbb{H}_1)^n)} = (2\pi)^{-2n} \int_{\mathbb{R}^n} \text{tr} \left( \hat{f}(\lambda)^* \hat{f}(\lambda) \right) |\lambda_1 \cdots \lambda_n| d\lambda. \tag{1.9}
\]

2. For suitable functions \( f \) on \( G \), say \( f \in \mathcal{S}(G) \),

\[
f(z, t) = (2\pi)^{-2n} \int_{\mathbb{R}^n} \text{tr} \left( \pi^\lambda_n(z, t)^* \hat{f}(\lambda) \right) |\lambda_1 \cdots \lambda_n| d\lambda. \tag{1.10}
\]

The sublaplacian on \( G \) is given by

\[
\mathcal{L} = \sum_{j=1}^n -(X_j^2 + Y_j^2), \tag{1.11}
\]

i.e. the direct sum of the sublaplacians \( \mathcal{L}_j = -(X_j^2 + Y_j^2) \) on each coordinate group.
of the product. By the above discussion, the eigenvalues of $\mathcal{L}$ are then the numbers
\begin{equation}
\sum_{j=1}^{n} (2\alpha_j + 1)|\lambda_j|, \quad \alpha_j \in \mathbb{N}, \; \lambda_j \in \mathbb{R}_*,
\end{equation}
and the expansion
\begin{equation}
f(z, t) = c'_n \sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}_*^n} f * \phi^n_{\alpha, \lambda}(z, t) |\lambda_1 \cdots \lambda_n| d\lambda
\end{equation}
holds for $f \in L^2(G)$, where the eigenfunctions $\phi^n_{\alpha, \lambda}$ are now given by
\begin{equation}
\phi^n_{\alpha, \lambda}(z, t) = (2\pi)^{-\frac{n}{2}} e^{-i\lambda \cdot t} \prod_{j=1}^{n} L_{\alpha_j} \left( \frac{1}{2} |\lambda_j| |z_j|^2 \right) e^{-\frac{1}{4} |\lambda_j|^2 |z_j|^2},
\end{equation}
for $\alpha \in \mathbb{N}^n, \lambda \in \mathbb{R}_n^*$. 

A function on $G$ is called polyradial if it only depends on the quantities $|z_1|, \ldots, |z_n|$, and $t$. Let $\mathcal{A}$ be the subalgebra of $L^1(G)$ of polyradial functions. The Gelfand transform of a function $f \in \mathcal{A}$ is given by
\begin{equation}
\hat{f}(\alpha, \lambda) = \int_{G} f(z, t) \bar{\phi^n_{\alpha, \lambda}(z, t)} dz dt, \quad \alpha \in \mathbb{N}^n, \lambda \in \mathbb{R}_n^*.
\end{equation}
We then have that, for appropriate functions $f$ and $g$ in $\mathcal{A}$,
\begin{equation}
\int_{G} f(z, t) \overline{g(z, t)} dz dt = (2\pi)^{-2n} \sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}_n^*} \hat{f}(\alpha, \lambda) \overline{\hat{g}(\alpha, \lambda)} |\lambda_1 \cdots \lambda_n| d\lambda.
\end{equation}
The proof of formula (1.16) follows as in the Heisenberg group case (see [Tha98]). We obtain from (1.16) the Plancherel formula
\begin{equation}
\int_{G} |f(z, t)|^2 dz dt = (2\pi)^{-2n} \sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}_n^*} |\hat{f}(\alpha, \lambda)|^2 |\lambda_1 \cdots \lambda_n| d\lambda
\end{equation}
and the inversion formula

\[ f(z, t) = (2\pi)^{-2n} \sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}^n} \hat{f}(\alpha, \lambda) \phi^{n}_{\alpha, \lambda}(z, t) |\lambda_1 \cdots \lambda_n| d\lambda. \] (1.18)

This implies that, if \( f \in \mathcal{S}(G) \cap \mathcal{A}, \)

\[(L_j f)\hat{\gamma}(\alpha, \lambda) = (2\alpha_j + 1)|\lambda_j| \hat{f}(\alpha, \lambda) \] (1.19a)

\[(L f)\hat{\gamma}(\alpha, \lambda) = \sum_{j=1}^{n} (2\alpha_j + 1)|\lambda_j| \hat{f}(\alpha, \lambda) \] (1.19b)

\[(-iT_j f)\hat{\gamma}(\alpha, \lambda) = \lambda_j \hat{f}(\alpha, \lambda), \] (1.19c)

for \( \alpha_j \in \mathbb{N} \) and \( \lambda_j \in \mathbb{R}_+. \)

On \( G = (\mathbb{H}_1)^n, \) the operators \( L_k \) and \(-iT_j, k, j = 1, \ldots, n, \) are essentially self-adjoint on \( \mathcal{S}(G) \) and commute with each other. Hence, by the spectral theorem, we can define the operator \( m(L_1, \ldots, L_n, -iT_1, \ldots, -iT_n) \) for any function \( m \in L^\infty(\mathbb{R}_+^n, \mathbb{R}^n). \) Moreover, by equations (1.19), we have that

\[(m(L_1, \ldots, L_n, -iT)f)\hat{\gamma}(\alpha, \lambda) = m(2\alpha_1 + 1)|\lambda_1|, \ldots, (2\alpha_n + 1)|\lambda_n|, \lambda) \hat{f}(\alpha, \lambda), \] (1.20)

for any proper polyradial function \( f, \) where \( T \) denotes the vector \((T_1, \ldots, T_n). \) We can also define, via the spectral theorem, the operator

\[ m(|T_1|^{-1}L_1, \ldots, |T_n|^{-1}L_n, -iT), \]

for which we then have

\[(m(|T_1|^{-1}L_1, \ldots, |T_n|^{-1}L_n, -iT)f)\hat{\gamma}(\alpha, \lambda) = m(2\alpha_1 + 1, \ldots, 2\alpha_n + 1, \lambda) \hat{f}(\alpha, \lambda). \] (1.21)
1.2 Littlewood-Paley Theory

In this section we develop a Littlewood-Paley theory on the group G. For this, we proceed as in [MRS96] and fix \( \phi \in C_0^\infty(\mathbb{R}) \), supported in \((1/2, 2)\), such that

\[
\sum_{j \in \mathbb{Z}} \phi(2^j |x|)^2 = 1, \quad x \neq 0.
\]  

Define then, for \( j \in \mathbb{Z} \) and \( l = (l_1, \ldots, l_n) \), the functions

\[
\phi_j(x) = \phi(2^{-j} x), \quad x \in \mathbb{R}_+, \\
\tilde{\phi}_l(y) = \phi(2^{-l_1} |y_1|) \cdots \phi(2^{-l_n} |y_n|), \quad y \in \mathbb{R}^n.
\]

Let the \( \varphi_j \) be the kernel of the operator \( \phi_j(L) \) on \( G \),

\[
\varphi_j = \phi_j(L)\delta_0,
\]

where \( \delta_0 \) is the Dirac-delta distribution centered at zero, and \( \psi_l \) the kernel of the operator \( \tilde{\phi}_l(-iT) \),

\[
\psi_l = \tilde{\phi}_l(-iT)\delta_0.
\]

Recall that \( T \) denotes \((T_1, \ldots, T_n)\). Note that

\[
\varphi_j * \psi_l = (\phi_j \otimes \tilde{\phi}_l)(L, -iT)\delta_0.
\]

We then define the \( g \)-function

\[
g_1(f)(z, t) = \left( \sum_{j,l} |f * (\varphi_j * \psi_l)(z, t)|^2 \right)^{1/2},
\]  

(1.23)

for, say, \( f \in S(G) \). The Plancherel formula, together with (1.22), implies that \( g_1 \) is an isometry on \( L^2(G) \).
Proposition 1.5. For $1 < p < \infty$, there exists $C_p > 0$ such that

$$C_p^{-1}||f||_{L^p} \leq ||g_1(f)||_{L^p} \leq C_p||f||_{L^p}. \quad (1.24)$$

Proof. The proof follows as the one for Proposition 4.1 in [MRS96]. \qed

We now formally define $\Phi_j$ as

$$\Phi_j = \phi_j\left(\sum_{k=1}^{n} |T_k|^{-1} L_k\right)\delta_0,$$

the kernel of the operator $\phi_j\left(\sum_{k=1}^{n} |T_k|^{-1} L_k\right)$. We then define the second $g$-function

$$g_2(f)(z,t) = \left(\sum_{j,l} |f \ast (\Phi_j \ast \psi_l)(z,t)|^2\right)^{1/2} \quad (1.25)$$

As above, $g_2$ is an isometry on $L^2(G)$, and we have the following proposition.

Proposition 1.6. For $1 < p < \infty$, there exists $C'_p > 0$ such that

$$C'_p^{-1}||f||_{L^p} \leq ||g_2(f)||_{L^p} \leq C'_p||f||_{L^p}. \quad (1.26)$$

Proof. The proof is this proposition is also similar to the one of Proposition 4.4 in [MRS96], except for the application of a proper version of Corollary 4.3 there. \qed

See Section 4.3 for details on the Littlewood-Paley theory on products of $H$-type groups, which include the particular case $(\mathbb{H}_1)^n$. 

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Chapter 2

A multiplier theorem on $({\mathbb H}_1)^n$

2.1 Sobolev spaces

Let $f \in l^2(\mathbb{Z})$. We define the operator $\Delta$ as

$$\Delta f(k) = f(k) - f(k-1), \quad k \in \mathbb{Z}.$$  

If $f$ has a sufficiently rapid decay at infinity, then

$$(\Delta f)^\wedge(\xi) = \sum_{k \in \mathbb{Z}} (f(k) - f(k-1)) e^{-2\pi ik\xi} = (1 - e^{-2\pi i \xi}) \hat{f}(k).$$

We can thus define the operator $(1 + |\Delta|)^\alpha, \alpha > 0$, via its Fourier transform by

$$((1 + |\Delta|)^\alpha f)^\wedge(\xi) = (1 + |1 - e^{-2\pi i \xi}|^\alpha) \hat{f}(\xi).$$

We define the discrete Sobolev space $l^2_\alpha(\mathbb{Z})$ as the set of functions $f$ on $\mathbb{Z}$ such that

$$(1 + |\Delta|)^\alpha f \in l^2(\mathbb{Z})$$
with norm

\[ \|f\|_{L^2_\alpha(Z)} = \|(1 + |\Delta|)^\alpha f\|_{L^2(Z)} = \left( \int_0^1 \left(1 + |1 - e^{-2\pi i \xi}|\right)^\alpha \hat{f}(\xi) \, d\xi \right)^{1/2}. \]

We will make use of the following proposition in the next section. Its proof can be found in [MRS96].

**Proposition 2.1.** Let \( f \in L^2_\alpha(\mathbb{R}) \), the standard Sobolev space of degree \( \alpha \), for \( \alpha > 1/2 \), and set \( g = f|_{\mathbb{Z}} \). Then \( g \in l^2_\alpha(\mathbb{Z}) \) and, moreover, for every \( R \geq \varepsilon > 0 \),

\[ \|(1 + |R\Delta|)^\alpha g\|_2 \leq C_\varepsilon \left( \int_{\mathbb{R}} \left(1 + |2\pi \xi R|\right)^\alpha \hat{f}(\xi) \, d\xi \right)^{1/2}, \quad (2.1) \]

where the constant \( C_\varepsilon \) depends only on \( \varepsilon \).

For \( \alpha > 0 \), we define the fractional differentiation operator \((1 + |\partial|)^\alpha\) on its Fourier transform by

\[ ((1 + |\partial|)^\alpha f) \hat{(}\xi) = (1 + |2\pi \xi|)^\alpha \hat{f}(\xi). \]

With this notation, equation (2.1) is written then as

\[ \|(1 + |R\Delta|)^\alpha g\|_2 \leq C_\varepsilon \|(1 + |R\partial|)^\alpha f\|_{L^2}. \]

From now on we fix \( \eta_0 \in C_0^\infty \) of compact support in \( \mathbb{R}_+ \), such that \( \eta_0 \geq 0 \), \( \eta_0(x) = 1 \) if \( x \in [1/2, 2] \) and \( \eta_0(x) = 0 \) if \( x \notin (1/4, 4) \). We then define the function \( \eta \) on \( \mathbb{R}^n \times \mathbb{R}^n \) by \( \eta(x, y) = \eta_1(x)\eta_2(y) \), where

\[ \eta_1(x) = \begin{cases} \eta_0(x_1 + \ldots + x_n) & x_1, \ldots, x_n > 0 \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \eta_2(y) = \eta_0(|y_1|) \cdots \eta_0(|y_n|). \]

(2.2)

Given an \((n+1)\)-tuple pair \( r = (r_0, r_1, \ldots, r_n) \) of positive numbers, we define the
functions, on $\mathbb{R}^n \times \mathbb{R}^n$,

$$\eta^r(x, y) = \eta(r_0 x, r_1 y_1, \ldots, r_n y_n)$$

$$\eta_r(x, y) = \eta(x/r_0, y_1/r_1, \ldots, y_n/r_n).$$

(2.3)

For $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ $n$-tuples of positive numbers, we define the multi-parameter scale invariant localized Sobolev space $\mathcal{L}_{\alpha, \beta, \text{loc}}$ as the set of functions $f$ on $\mathbb{Z}^n \times \mathbb{R}^n$ such that the quantity

$$||f||_{\mathcal{L}_{\alpha, \beta, \text{loc}}} = \sup_{r \in \mathbb{R}_{+}^{n+1}} \left( \prod_{j=1}^{n} (1 + r_0 |\Delta_j|)^{\alpha_j} (1 + r_0 |\Delta_j| + |r_j \partial_j|)^{\beta_j} f \eta_r \right)_{l^2(L^2)}$$

(2.4)

is finite. Here, $|| \cdot ||_{l^2(L^2)}$ denotes the norm of the Hilbert space $l^2(\mathbb{Z}^n, L^2(\mathbb{R}^n))$ of functions on $\mathbb{Z}^n \times \mathbb{R}^n$, and $\Delta_j$ and $\partial_j$ denote the partial difference and differential operators, on the $j$th discrete and the $j$th continuous variables, respectively.

Note that the definition above of the spaces $\mathcal{L}_{\alpha, \beta, \text{loc}}$, as well as the results below, does not depend on the specific choice of the function $\eta$.

We now proceed to study some properties of the space $\mathcal{L}_{\alpha, \beta, \text{loc}}$ that will be of use in the next chapters. First, since

$$(1 + r_0 x_j)^{\alpha_j} (1 + |r_0 x|_1 + |r_j y_j|)^{\beta_j} \leq \left( 1 + |r_0 x|_1 + \sum_{j=1}^{n} |r_j y_j| \right)^{\alpha_j + \beta_j},$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ and $|x|_1 = x_1 + \ldots + x_n$, then

$$l^2(\mathbb{Z}^n, L^2(\mathbb{R}^n))_{\alpha_1 + |\beta|_1, \text{loc}} \subset \mathcal{L}_{\alpha, \beta, \text{loc}},$$

where, for $s > 0$, $l^2(\mathbb{Z}^n, L^2(\mathbb{R}^n))_{s, \text{loc}}$ denotes the scale invariant localized Sobolev space
with norm
\[
\sup_{r \in \mathbb{R}^{n+1}_+} (r_0^n r_1 \cdots r_n)^{-1/2} \left\| (1 + \sum_{j=1}^n (|r_0 \Delta_j| + |r_j \partial_j|))^{s} f_{\eta_r} \right\|_{l^2(L^2)}.
\]

Note that the inclusion is continuous, i.e.
\[
\|f\|_{L_{\alpha,\beta,\text{loc}}} \leq \|f\|_{t^2(\mathbb{Z}^n,L^2(\mathbb{R}^n))_{|\alpha|+|\beta|,\text{loc}}}.
\]

**Proposition 2.2.** Suppose \(f(k,x) = F(k)\), where \(F\) is a function on \(\mathbb{Z}^n\) such that \(F \in l^2_{|\alpha|+|\beta|,\text{loc}}, \text{ i.e.} \)
\[
\sup_{r_0 > 0} r_0^{-n/2} \left\| (1 + |r_0 \Delta|)^{|\alpha|+|\beta|} f_{\eta_1,0} \right\|_{l^2} < \infty.
\]

Then \(f \in L_{\alpha,\beta,\text{loc}}\).

Here \(\eta_{1,r_0}\) is defined, for \(r_0 \in \mathbb{R}_+\), analogously as \(\eta_r\) by \(\eta_{1,r_0}(x) = \eta_1(x/r_0)\), and \(|\Delta|\) denotes \(|\Delta_1| + \ldots + |\Delta_n|\).

**Proof.** This proposition follows easily from the observation, for \((\xi,\zeta) \in [0,1]^n \times \mathbb{R}^n\),
\[
\hat{f}_{\eta_r}(\xi,\zeta) = \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{R}^n} f_{\eta_r}(k,x)e^{-2\pi ik \cdot \xi}e^{-2\pi ix \cdot \zeta} dx
\]
\[
= \hat{F}_{\eta_1,0}(\xi) r_1 \cdots r_n \hat{\eta_0}(r_1\zeta_1) \cdots \hat{\eta_0}(r_n\zeta_n),
\]
and, for each \(j\),
\[
\int_{\mathbb{R}} \left| (1 + R + r_j |2\pi \zeta_j|)^{\beta_j} r_j \hat{\eta_0}(r_j \zeta_j) \right|^2 d\zeta_j \lesssim r_j (1 + R)^{2\beta_j},
\]
for any \(R > 0\).
Proposition 2.3. Let \( f \) be a function on \( \mathbb{Z} \) supported on \( \mathbb{N} \), and define, for \( k \in \mathbb{Z}^n \),
\[
F(k) = \begin{cases} 
 f(k_1 + \ldots + k_n) & \text{if } k_j \geq 0, \, j = 1, \ldots, n \\
0 & \text{otherwise.}
\end{cases}
\]
Then, if \( f \in l^2_{|\alpha|,\text{loc}}(\mathbb{Z}) \), \( F \in l^2_{\alpha,\text{loc}}(\mathbb{Z}^n) \), for \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n \).

Proof. This proposition follows by induction on \( n \), with the first step as in Proposition 2.4 in [MS94].

2.2 Multiplier theorem

Let \( m \) be a bounded continuous function defined on the set
\[
(2N + 1)^n \times \mathbb{R}^n = \{(2\alpha_1 + 1, \ldots, 2\alpha_n + 1, \lambda_1, \ldots, \lambda_n) : \alpha_j \in \mathbb{N}, \lambda_j \in \mathbb{R}\}.
\]
Then, we can define the operator
\[
T = m(|T_1|^{-1}L_1, \ldots, |T_n|^{-1}L_n, -iT)
\]
by the spectral theorem.

Define \( \tilde{m} \) on \( \mathbb{Z}^n \times \mathbb{R}^n \) by
\[
\tilde{m}(\alpha, \lambda) = \begin{cases} 
 m(2\alpha_1 + 1, \ldots, 2\alpha_n + 1, \lambda_1, \ldots, \lambda_n) & \text{if } \alpha_k \geq 0, 1 \leq k \leq n \\
0 & \text{otherwise.}
\end{cases}
\]

We have the following theorem.

Theorem 2.4. Suppose that, for some \( s_k > 1, s'_k > 1/2, k = 1, \ldots, n \), the function \( \tilde{m} \)
is in \( L_{s,s',\text{loc}} \), where \( s = (s_1, \ldots, s_n) \) and \( s' = (s'_1, \ldots, s'_n) \). Then, the operator (2.5) is
bounded in $L^p(G)$, with norm

$$||T||_{L^p ightarrow L^p} \lesssim ||\tilde{m}||_{L^{s,s',\text{loc}}}.$$ (2.6)

The constant in (2.6) depends only on $n$, $s$, $s'$, and $p$.

Note that Theorem 2.4 is not a “tensor-product” version of Theorem 2.3 in [MRS96], although it is just slightly stronger. The difference is that, here, we only require scale invariance over $|\alpha|$, and not on each variable $\alpha_k$.

Given two $n$-tuples $s$ and $s'$ of positive numbers, we say that a function $f$ on $\mathbb{R}_+^n \times \mathbb{R}^n$ is in the space $L^{s,s',\text{loc}}$ if

$$||f||_{L^{s,s',\text{loc}}} = \sup_{r \in \mathbb{R}_+^{n+1}} ||f^r \eta||_{L^2(L_n^2)},$$

where

$$f^r(x, y) = f(r_0 x, r_1 y_1, \ldots, r_n y_n)$$

and

$$||f||_{L^2(L_n^2)^2} = \int_{\mathbb{R}_+^n \times \mathbb{R}^n} \prod_{k=1}^n \left( 1 + |2\pi \xi_k|^2 ight)^{s_k} \left( 1 + |2\pi \zeta_k| + |2\pi \xi_k| \right)^{s_k'} \hat{f}(\xi, \zeta)^2 d\xi d\zeta.$$

We now have the following two corollaries.

**Corollary 2.5.** Let $m$ be a bounded continuous function on $\mathbb{R}_+^n \times \mathbb{R}^n$ and suppose that, for some $s_k > 1, s'_k > 1/2, k = 1, \ldots, n$, $m \in L^{s,s',\text{loc}}$. Then the operators $T$, given by (2.5), and

$$\mathcal{M} = m(L_1, \ldots, L_n, -iT_1, \ldots, -iT_n),$$

are
are bounded in $L^p(G)$, $1 < p < \infty$, with norms

$$
||T||_{L^p \to L^p} \lesssim ||m||_{\mathcal{L}_{s,s',\text{loc}}}
$$

$$
||M||_{L^p \to L^p} \lesssim ||m||_{\mathcal{L}_{s,s',\text{loc}}}.
$$

**Corollary 2.6.** Let $m$ be a bounded continuous function on $\mathbb{R}$, such that, for some $s_0 > 3n/2$, $m \in L^2_{s_0,\text{loc}}(\mathbb{R})$. Then $m$ is a multiplier, i.e. the operator $m(\mathcal{L})$ is bounded in $L^p(G)$.

These two Corollaries follow from Theorem 2.4, application of Propositions 2.1, 2.2, and 2.3, and the fact that, if we set

$$
f(x, y) = g(x_1|y_1|, \ldots, x_n|y_n|, y),
$$

then $g \in \mathcal{L}_{s,s',\text{loc}}$ implies $f \in \mathcal{L}_{s,s',\text{loc}}$ (perhaps with another bump function $\eta$). Indeed, note that

$$
\partial_{x_j} (f^r(x, y)\eta(x, y)) = r_0|y_j|(\partial_{x_j} g)^r(x_1|y_1|, \ldots, x_n|y_n|, y)\eta(x, y)
$$

$$
+ g^r(x_1|y_1|, \ldots, x_n|y_n|, y)(\partial_{x_j} \eta)(x, y),
$$

which implies that

$$
||\partial_{x_j}(f^r \eta)||_{L^2} \leq \left( \int |r_0|y_j|(\partial_{x_j} g)^r(x_1|y_1|, \ldots, x_n|y_n|, y)\eta(x, y)|^2 dx dy \right)^{1/2}
$$

$$
+ \left( \int |g^r(x_1|y_1|, \ldots, x_n|y_n|, y)(\partial_{x_j} \eta)(x, y)|^2 dx dy \right)^{1/2}.
$$

Since, on $\text{supp} \eta$, $|y_k| \sim 1$ for $k = 1, \ldots, n$, we can make the substitution $x_k \mapsto \tilde{x}_k/|y_k|$ to obtain

$$
||\partial_{x_j}(f^r \eta)||_{L^2} \leq ||r_0(\partial_{x_j} g)^r \eta||_{L^2} + ||g \cdot \partial_{x_j} \eta||_{L^2} \leq 2||\partial_{x_j}(g^r \eta)||_{L^2},
$$

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where
\[ \bar{\eta}(x, y) = \eta\left(\frac{x_1}{|y_1|} + \ldots + \frac{x_n}{|y_n|}, y\right). \]

Note that, since supp \( \bar{\eta} \), on the \( x \)-coordinate, is comparable to supp \( \eta \) (since \( |y_k| \sim 1 \)), we see that, taking the supremum over \( r \in \mathbb{R}^{n+1}_+ \)
\[ \|f\|_{L_{s,s}',\text{loc}} \lesssim \|g\|_{L_{s,s}',\text{loc}}. \]

Similarly, we can prove the relation
\[ \|f\|_{L_{s,s},\text{loc}} \lesssim \|g\|_{L_{s,s},\text{loc}} \]
for general \( n \)-tuples \( s \) and \( s' \) of nonnegative integers, and the estimate for \( s, s' \in \mathbb{R}^n_+ \) follows by interpolation. See the proofs of Theorem 5.1 and Corollaries 5.2 and 5.3 for the details in the more general case of \( H \)-type groups.

We now proceed to prove Theorem 2.4.

### 2.2.1 Proof of Theorem 2.4

Recall that \( \eta_0 \in C_0^\infty \), \( \eta_0 \geq 0 \), supported in \((1/4, 4)\), and with \( \eta_0 \equiv 1 \) on \([1/2, 2]\). We then define, for \( j \in \mathbb{Z} \), \( l \in \mathbb{Z}^n \),
\[ m_{j,l}(x, y) = m(x, y)\eta_{(j,l)}(x, y), \]
for \( x \in \mathbb{R}^n_+, y \in \mathbb{R}^n \), where \( \eta_r \) is defined as in (2.3) and \( r(j, l) \) is the \((n + 1)\)-tuple \((2^j, 2^{l_1}, \ldots, 2^{l_n})\). Set \( M_{j,l} \) to be the kernel of the operator
\[ m_{j,l}(|T_1|^{-1}\mathcal{L}_1, \ldots, |T_n|^{-1}\mathcal{L}_n, -iT_1, \ldots, -iT_n). \]

As in [MRS96], the proof of Theorem 2.4 will follow from the following crucial Lemma.
Lemma 2.7. Under the hypotheses of Theorem 2.4, for $j \in \mathbb{Z}, l \in \mathbb{Z}^n$, we have the estimate

$$
\int_G \left| \prod_{1 \leq k \leq n} (1 + 2^{j+k} |z_k|^2)(1 + 2^{l_k} |t_k|) \right|^2 |M_{j,l}(z,t)|^2 dzdt \lesssim 2^{nj+2l}
$$

$$
\times 2^{-nj-l} \sum_{\alpha \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \left| \prod_{1 \leq k \leq n} (1 + 2^j |\Delta_k|^2)(1 + 2^j |\Delta_k| + 2^l |\partial_k|) \right|^2 |\tilde{m}_{j,l}(\alpha, \lambda)|^2 d\lambda, 
$$

(2.7)

where $\bar{l} = l_1 + \ldots + l_n$ and the constant of inequality (2.7) is independent of $m, j$, and $l$.

The function $\tilde{m}_{j,l}$ on $\mathbb{Z}^n \times \mathbb{R}^n$ is defined as $\tilde{m}$ above:

$$
\tilde{m}_{j,l}(\alpha, \lambda) = \begin{cases} 
m_{j,l}(2\alpha_1 + 1, \ldots, 2\alpha_n + 1, \lambda) & \text{if } \alpha_1, \ldots, \alpha_n \geq 0, \\
0 & \text{otherwise}. 
\end{cases}
$$

Before proving Lemma 2.7, we first observe that it implies Theorem 2.4. By interpolation with

$$
\int_G \left| M_{j,l}(z,t) \right|^2 dzdt \lesssim 2^{nj+2l} 2^{-(nj+l)} \sum_{\alpha \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |\tilde{m}_{j,l}(\alpha, \lambda)|^2 d\lambda,
$$

which follow by the Plancherel formula (note that $|\lambda_k| \sim 2^k$ on the support of $m_{j,l}$), we get the estimate, for $0 \leq \varepsilon \leq 1/2$,

$$
\int_G \left| \prod_{1 \leq k \leq n} (1 + 2^{j+k} |z_k|^2)^{1/2+\varepsilon}(1 + 2^{l_k} |t_k|)^{1/2+\varepsilon} M_{j,l}(z,t) \right|^2 dzdt
\lesssim 2^{nj+2l} \left| \tilde{m} \right|_{\mathcal{L}_{\gamma,\delta,\text{loc}}},
$$

(2.8)

where $\gamma = (1 + 2\varepsilon, \ldots, 1 + 2\varepsilon)$ and $\delta = (1/2 + \varepsilon, \ldots, 1/2 + \varepsilon)$.

For $f \in L^p(G)$, Proposition 1.6 and the fact that $\eta_0 \equiv 1$ on the support of $\phi$ imply
that
\[ ||Tf||_{L^p} \leq C_p ||g_2(Tf)||_{L^p} = C_p \left( \sum_{(j,l) \in \mathbb{Z} \times \mathbb{Z}^n} |f_{j,l} \ast M_{j,l}|^2 \right)^{1/2} \]

where \( f_{j,l} = f \ast (\Phi_j \ast \psi_l) \). By the Cauchy-Schwartz inequality, and using (2.8),
\[
|f_{j,l} \ast M_{j,l}(z,t)|^2 \lesssim ||\tilde{m}||_{L^2_{\gamma,\delta,loc}}^2 \int_G \frac{\left|f_{j,l}((z,t)(w,u)^{-1})\right|^2}{W(w,u)} dwdu,
\]

where
\[
W(w,u) = 2^{-nj-2l} \prod_{1 \leq k \leq n} (1 + 2^{j+l_k}|w_k|^2)^{1+2\varepsilon}(1 + 2^k|u_k|)^{1+2\varepsilon}.
\]

Without loss of generality, we can assume \( p \geq 2 \) (the case \( p \leq 2 \) follows by duality).

Then, there exists \( g \in L^{(p/2)'}(G) \), with \( g \geq 0 \) and \( ||g||_{L^{(p/2)'}(G)} = 1 \), such that
\[
\left( \sum_{(j,l) \in \mathbb{Z} \times \mathbb{Z}^n} |f_{j,l} \ast M_{j,l}|^2 \right)^{1/2} \leq 2 \int_G \sum_{(j,l) \in \mathbb{Z} \times \mathbb{Z}^n} |f_{j,l} \ast M_{j,l}(z,t)|^2 g(z,t) dzdt.
\]

Therefore, Fubini’s theorem and (2.9) imply
\[
||Tf||_{L^p}^2 \lesssim ||\tilde{m}||_{L^2_{\gamma,\delta,loc}}^2 \int_G \left( \sum_{j,l} |f_{j,l}(z,t)|^2 \right) \left( \int_G \frac{g(z,t)}{W((w,u)^{-1}(z,t))} dzdt \right) dwdu.
\]

Set
\[
A_{p_k} = \begin{cases} 
\{ z \in \mathbb{C} : |z_k|^2 \leq 2^{-j-l_k} \} & \text{if } p_k = 0 \\
\{ z \in \mathbb{C} : 2^{-j-l_k+p_k-1} < |z_k|^2 \leq 2^{-j-l_k+p_k} \} & \text{if } p_k \geq 1,
\end{cases}
\]

for \( p_1, \ldots, p_n \in \mathbb{N} \), and, for \( q_1, \ldots, q_n \in \mathbb{N} \),
\[
B_{q_k} = \begin{cases} 
\{ t \in \mathbb{R} : |t| \leq 2^{-l_k} \} & \text{if } q_k = 0 \\
\{ t \in \mathbb{R} : 2^{-l_k+q_k-1} < |t| \leq 2^{-l_k+q_k} \} & \text{if } q_k \geq 1.
\end{cases}
\]
Thus, on $A_{p_1} \times \ldots \times A_{p_n} \times B_{q_1} \times \ldots \times B_{q_n}$,

$$W(w, u) \geq 2^{-nj - 2l} \prod_{1 \leq k \leq n} (1 + 2^{p_k - 1}1 + 2^{q_k - 1})^{1 + 2 \varepsilon} 
\geq \prod_{1 \leq k \leq n} 2^{-j - l_k + p_k} \cdot 2^{2p_k \varepsilon} \cdot 2^{-l_k + q_k} \cdot 2^{2q_k \varepsilon}.$$ 

Hence,

$$\int_G \frac{g((w, u)(z, t))}{W(w, u)} \, dwdu \leq \sum_{(p, q) \in \mathbb{N}^n \times \mathbb{N}^n} \int_{A_{p_1}} \ldots \int_{A_{p_n}} \int_{B_{q_1}} \ldots \int_{B_{q_n}} \prod_{1 \leq k \leq n} 2^{-j - l_k + p_k} \cdot 2^{2p_k \varepsilon} \cdot 2^{-l_k + q_k} \cdot 2^{2q_k \varepsilon} \mathcal{M}(g)(z, t) = C_\varepsilon \mathcal{M}(g)(z, t),$$

where $\mathcal{M}$ denotes the strong maximal function

$$\mathcal{M}(g)(z, t) = \sup_{r, r' \in \mathbb{R}^n} \frac{1}{r_1^2 \cdots r_n^2} \int_{r_1} \ldots \int_{r_n} |g((w, u)(z, t))| \, dwdu. \quad (2.11)$$

(See [Chr92].) Therefore, (2.10) is majorized by

$$||\tilde{m}||_{L^2_{\gamma, \delta, \text{loc}}} \int_G (g_2(f)(z, t))^2 \cdot \mathcal{M}(g)(z, t) \, dzdt \leq ||\tilde{m}||_{L^2_{\gamma, \delta, \text{loc}}}^2 ||g_2(f)||_{L^p}^2 \mathcal{M}(g) ||_{L^p/(p/2)},$$

by Hölder’s inequality. Theorem 2.4 now follows by Proposition 1.6 and Theorem 2.3 in [Chr92], which states that $\mathcal{M}$ is a bounded operator on $L^p(G)$, for $1 < p \leq \infty$.

**Proof of Lemma 2.7:** We now turn to the proof of Lemma 2.7. First, since each of the operators $|T_k|^{-1} \mathcal{L}_k$ and $-iT_k$ are homogeneous of order 0 and 2, respectively, we have that

$$2^{2f} M_{j,l}(z_1/2^{l_1/2}, \ldots, z_n/2^{l_n/2}, t_1/2^{l_1}, \ldots, t_n/2^{l_n}) = \tilde{M}(z, t),$$

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where $\bar{M}$ is the kernel of the operator

$$m_{j,l}(|T_1|^{-1}L_k, \ldots, |T_k|^{-1}L_k, -i2^l T_1, \ldots, -i2^n T_n).$$

Thus, by rescaling, it follows that it is enough to consider the case $l_k = 0$, $k = 1, \ldots, n$.

We will simply write $M$ and $m$ (respectively $\tilde{m}$) for $M_{j,0}$ and $m_{j,0}$ (respectively $\tilde{m}_{j,0}$).

Thus, we will prove the estimate

$$\int_G \left| \prod_{1 \leq k \leq n} (1 + 2^j |z_k|^2)(1 + |t_k|)M(z,t) \right|^2 dz dt \lesssim \sum_{\alpha \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \left| \prod_{1 \leq k \leq n} (1 + 2^j |\Delta_j|)^2(1 + 2^j |\Delta_j| + |\partial_k|)\tilde{m}(\alpha, \lambda) \right|^2 d\lambda. \quad (2.12)$$

It is not hard to see that the leading term of the left hand side of (2.12) is

$$\int_G \left| \prod_{1 \leq k \leq n} (2^j |z_k|)|t_k|M(z,t) \right|^2 dz dt. \quad (2.13)$$

By the inversion formula $G$, (1.18), and well-know properties of Laguerre polynomials [Tha93], we see that the Gelfand transforms of the functions $|z_k|^2 F(z,t)$ and $i t_k F(z,t)$, for a radial function $F$, are given by

$$\frac{2}{|\lambda_k|} (\alpha_k \Delta_k - \tau_k^1 \alpha_k \Delta_k) f(\alpha, \lambda), \quad (2.14a)$$

and

$$\partial_k f(\alpha, \lambda) + \frac{1}{2\lambda_k} (-\alpha_k \Delta_k - \tau_k^1 \alpha_k \Delta_k) f(\alpha, \lambda), \quad (2.14b)$$

respectively, where $\tau_k^p$ denotes the operator

$$\tau_k^p f(\alpha, \lambda) = f(\alpha_1, \ldots, \alpha_k + p, \ldots, \alpha_n, \lambda).$$
If we define $\partial_{|z_k|^2}$ and $\partial_{it_k}$ to be the operators given by the equations (2.14), respectively, then we have to estimate, by the Plancherel formula (1.17),

$$
\sum_{\alpha \in \mathbb{N}_n} \int_{\mathbb{R}^n} \prod_{1 \leq k \leq n} (2^j \partial_{|z_k|^2})(\partial_{it_k}) \tilde{m}(\alpha, \lambda) \left| \lambda_1 \right| \cdots \left| \lambda_n \right| d\lambda.
$$

(2.15)

First, observe that the operators $\partial_{|z_k|^2}$ and $\partial_{it_k}$ can be rewritten as

$$
\partial_{|z_k|^2} = -\frac{2}{|\lambda_k|} (\tau_1^{\lambda_k} \Delta_k^2 + \Delta_k),
$$

and

$$
\partial_{it_k} = \partial_k - \frac{1}{2\lambda_k} (\tau_0^{\lambda_k} + \tau_1^{\lambda_k}) \alpha_k \Delta_k.
$$

Heuristically, the contributions of each of the operators $\partial_{|z_k|^2}$ and $\partial_{it_k}$ on (2.15) are

$$
|\partial_{|z_k|^2}| \sim \frac{1}{|\lambda_k|} |\alpha_k| |\Delta_k|^2 \quad \text{and} \quad |\partial_{it_k}| \sim \frac{1}{|\lambda_k|} (|\lambda_k| \partial_k + |\alpha_k| |\Delta_k|).
$$

Thus, since $|\alpha| \sim 2^j$ and $|\lambda_k| \sim 1$ on the support of $\tilde{m}$, we have that, formally,

$$
\left| \prod_{1 \leq k \leq n} (2^j \partial_{|z_k|^2})(\partial_{it_k}) \tilde{m} \right| \lesssim \prod_{1 \leq k \leq n} |2^j \Delta_k|^2 (|\partial_k| + 2^j |\Delta_k|) \tilde{m},
$$

which is the desired estimate.

To make things precise, note that

$$
\partial_{|z_k|^2} \partial_{it_k} = \left( -\frac{2}{|\lambda_k|} (\tau_1^{\lambda_k} \Delta_k^2 + \Delta_k) \right) \left( \partial_k - \frac{1}{2\lambda_k} (\tau_0^{\lambda_k} + \tau_1^{\lambda_k}) \alpha_k \Delta_k \right)
$$

$$
= -\frac{2}{|\lambda_k|} \tau_1^{\lambda_k} \Delta_k \partial_k - \frac{2}{|\lambda_k|} \Delta_k \partial_k + \frac{\tau_0^{\lambda_k} + \tau_1^{\lambda_k}}{|\lambda_k| \lambda_k} \Delta_k^2 
$$

$$
+ \frac{3\tau_0^{\lambda_k} + 2\tau_1^{\lambda_k} + \tau_2^{\lambda_k}}{|\lambda_k| \lambda_k} \alpha_k \Delta_k^2 + \frac{2\tau_0^{\lambda_k} + \tau_1^{\lambda_k} - \tau_2^{\lambda_k}}{|\lambda_k| \lambda_k} \Delta_k.
$$
where we have used the commutation relations

\[ \Delta_k \alpha_k = \alpha_k \Delta_k + \tau_k^{-1} \quad \text{and} \quad \alpha_k \tau_k^p = \tau_k^p \alpha_k - p\tau_k^p. \]

By the Plancherel formula, we obtain

\[
\int_G \left| \prod_{1 \leq k \leq n} (2^j |z_k|) |t_k| M(z, t) \right|^2 dz dt \lesssim 
\sum_{\alpha \in \mathbb{N}^n} \sum_{k=1}^n \sum_{0 \leq \mu_k \leq 2} \sum_{0 \leq \nu_k \leq 1} \int_{\mathbb{R}^n} \left| \prod_{1 \leq k \leq n} |\lambda_k|^{-2+\nu_k} 2^j \alpha_k^{\mu_k} \Delta_k^{\mu_k+1} \partial_k^{\nu_k} \tilde{m}(\alpha, \lambda) \right|^2 |\lambda_1| \cdots |\lambda_n| d\lambda. \quad (2.16)
\]

Now, on the support of \( \tilde{m} \),

\[ \alpha_k \leq |\alpha|_1 \sim 2^j; \quad |\lambda_k| \sim 1, \]

for \( k = 1 \ldots, n \). Thus, (2.16) is estimated by

\[
\sum_{\alpha \in \mathbb{N}^n} \sum_{k=1}^n \sum_{0 \leq \mu_k \leq 2} \sum_{0 \leq \nu_k \leq 1} \int_{\mathbb{R}^n} \left| \prod_{1 \leq k \leq n} (2^j \Delta_k)^{\mu_k+1} \partial_k^{\nu_k} \tilde{m}(\alpha, \lambda) \right|^2 d\lambda. \quad (2.17)
\]

Finally, the Plancherel theorem on \( \mathbb{Z}^n \times \mathbb{R}^n \), together with the basic inequality

\[ \sum_{1 \leq \mu, \nu \leq 3, 0 \leq \rho \leq 1} r^\mu r'^\nu \leq 2(1+r)^2(1+r+r'), \quad r, r' > 0, \]

imply the Lemma. \( \square \)
Chapter 3

Central quotients and projection operators

3.1 Central quotients

Let $A_1, \ldots, A_n$ be $n$ nonzero $d$-vectors which span $\mathbb{R}^d$, and let $A$ be the $d \times n$ matrix whose columns are the vectors $A_1, \ldots, A_n$. Then $A$ has rank $d$. We define the Lie group $G_A$ as the set $\mathbb{C}^n \times \mathbb{R}^d$ with operation

$$ (z, t) \cdot (w, s) = \left( z + w, t + s + \frac{1}{2} \Im \left( \sum_{k=1}^{n} z_k \bar{w}_k A_k \right) \right), $$

for $z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in \mathbb{C}^n, t, s \in \mathbb{R}^d, \bar{r}$ the complex conjugate of the number $r$, and $\Im(r)$ the imaginary part of $r$.

Examples

1. If $A_1 = e_1, \ldots, A_n = e_n$ are the standard basis elements for $\mathbb{R}^n$, then $A = I_n$, the identity matrix, and $G_A = (\mathbb{H}_1)^n$, i.e. the $n$-fold product of the Heisenberg group $\mathbb{H}_1$ of dimension $3$.

2. If $A_k = (1), 1 \leq k \leq n$, then $A = (1 \ldots 1)$ and $G_A$ is the Heisenberg group $\mathbb{H}_n$ of
3. If $A_k = a_k e_j$ for some $1 \leq j_k \leq d$, $a_k \in \mathbb{R}$, where the $e_j$ are the standard basis elements for $\mathbb{R}^d$, then $G_A$ is isomorphic to a product of Heisenberg groups. Indeed,

$$G_A \cong \mathbb{H}_{n_1} \times \mathbb{H}_{n_2} \times \ldots \times \mathbb{H}_{n_d},$$

where $n_j$ is the number of vectors $A_k$ which are equal to a multiple of $e_j$.

4. The simplest example of a group $G_A$ that is not isomorphic to a product of Heisenberg groups, is the group $G_A$ with $A$ given by

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$  

In general, the group $G_A$ has dimension $D = 2n + d$ and, since the vectors $A_1, \ldots, A_n$ are nonzero and span $\mathbb{R}^d$, the center of $G_A$, $\{(0, t) : t \in \mathbb{R}^d\}$ is isomorphic to $\mathbb{R}^d$. Moreover, $G_A$ is isomorphic to the quotient of $(\mathbb{H}_1)^n$ over some center subgroup, indeed, $G_A \cong (\mathbb{H}_1)^n/K$ where $K$ is the kernel of the map $\left((z,x) \mapsto (z,Ax)\right)$ from $\mathbb{C}^n \times \mathbb{R}^n$ onto $\mathbb{C}^n \times \mathbb{R}^d$ ($A$ has maximal rank). To see this note that the operation on $G_A$ is given by

$$(z,t) \cdot (w,s) = I \otimes A((z,x) \cdot_{(\mathbb{H}_1)^n} (w,y)),$$

where $I$ is the identity map on $\mathbb{C}^n$ and $x, y \in \mathbb{R}^n$ are such that $Ax = t$ and $Ay = s$. Also note that $K$ does not contain any of the individual centers of the product $(\mathbb{H}_1)^n$.

The converse is true. Let $K$ be a center subgroup of $(\mathbb{H}_1)^n$, not containing any of the individual centers. Then $K$ can be seen as a subspace of $\mathbb{R}^n$, say of dimension $p$. Let $v_1, \ldots, v_p$ be a set of generators for $K$, which we can assume (by changing the order of the factors of $(\mathbb{H}_1)^n$ if necessary) are of the form $v_j = (u_j,0,\ldots,-1,\ldots,0)$, where the $u_j$ are $(n-p)$-vectors and the $-1$ is in the $(n-p+j)$th position. Then $K$
is the null set of the $d \times n$ matrix $A$ given by

$$A = \begin{pmatrix} e_1 \ldots e_d & u_1 \ldots u_p \end{pmatrix},$$

where $d = n - p$ and the vectors $e_j$ are the standard basis vectors in $\mathbb{R}^d$. Therefore we conclude that $(\mathbb{H}_1)^n/K \cong G_A$.

From this observation one also concludes that $G_A \cong G_B$ if the matrices $A$ and $B$ are row equivalent.

The groups $G_A$ appear in the study of several complex variables, as the translation groups of certain submanifolds of $\mathbb{C}^{n+d}$ (see [NRS01]). Indeed, if we let $A(z)$ to be the $d$-vector

$$A(z) = \sum_{k=1}^{n} |z_k|^2 A_k,$$

and $\Sigma_A$ to be the manifold

$$\Sigma_A = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^d : \Im(w) = A(z)\},$$

then the group $G_A$ is isomorphic to the translation group of $\Sigma_A$. See Chapter 4 of [NRS01] for the details. Note that, in particular, in the case of the Heisenberg group the function $A(z)$ is equal to the squared norm of the vector $z$, so $\mathbb{H}_n$ is the translation group of the manifold

$$\{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \Im(w) = \sum_{k=1}^{n} |z_k|^2\},$$

the boundary of the $(n+1)$-dimensional Siegel space $\mathcal{U}_n$, as is well known (see [Ste93], Chapter XII).

The groups $G_A$ are homogeneous groups with dilations

$$\delta_r(z, t) = (rz, r^2 t), \quad r > 0,$$

(3.3)
and homogeneous norm
\[
|(z, t)| = (|z|^4 + |t|^2)^{1/4},
\]  
where $|z|$ and $|t|$ are the ordinary Euclidean norms of $z$ and $t$ in $\mathbb{C}^n$ and $\mathbb{R}^d$, respectively. It is easy to see that the dilations (3.3) are automorphisms and
\[
|\delta_r(z, t)| = r|(z, t)|
\]
for every $r > 0$. With such dilation structure, each group has homogeneous dimension $Q = 2n + 2d$ [FS82].

Haar measure on $G_A$ is given by Lebesgue measure $dzdt$ on $\mathbb{C}^n \times \mathbb{R}^d$. Note that this measure is homogeneous of order $Q$, i.e.
\[
\int_{G_A} f dzdt = r^Q \int_{G_A} f \circ \delta_r dzdt, \quad r > 0,
\]
for each $f \in L^1(G_A)$.

The Lie algebra $g_A$ of the group $G_A$ has basis $X_1, Y_1, \ldots, X_n, Y_n, T_1, \ldots, T_d$ defined by
\[
\begin{align*}
X_k &= \frac{\partial}{\partial x_k} + \frac{1}{2} y_k A_k \cdot \nabla \quad k = 1, \ldots, n \\
Y_k &= \frac{\partial}{\partial y_k} - \frac{1}{2} x_k A_k \cdot \nabla \quad k = 1, \ldots, n \\
T_j &= \frac{\partial}{\partial t_j} \quad j = 1, \ldots, d.
\end{align*}
\]  
(3.5)

Here we view each $z_k$ as $(x_k, y_k)$, and $\nabla$ is the operator
\[
\left( \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_d} \right) = (T_1, \ldots, T_d).
The commutator operations are given by

\[ [X_k, Y_k] = -A_k \cdot \nabla, \tag{3.6} \]

with all other commutators relations equal to zero. Note that each of the vector fields \( X_k \) and \( Y_k \) are \textit{homogeneous of order} 1, in the sense that

\[ X_k(f \circ \delta_r) = r(X_k f) \circ \delta_r, \tag{3.7} \]

where \( \delta_r \) is the dilation operation given by (3.3).

The Lie algebra \( g_A \) is stratified since

\[ g_A = V_1 \oplus V_2, \]

where \( V_1 = \text{span}\{X_1, Y_1, \ldots, X_n, Y_n\} \), \( V_2 = \text{span}\{T_1, \ldots, T_d\} \), and \( V_1 \) generates \( g_A \) as a Lie algebra. Here the symbol \( \oplus \) denotes vector space direct product. We define the sublaplacian \( \mathcal{L} \) as the operator

\[ \mathcal{L} = - \sum_{k=1}^{n} (X_k^2 + Y_k^2). \tag{3.8} \]

The Fourier analysis of the groups \( G_A \) has been worked out in [NRS01], and we present a summary here. A set of irreducible unitary representations is given explicitly by the operators \( \pi^A_{\sigma} \), for \( \sigma \in \mathbb{R}^d \),

\[ \pi^A_{\sigma}(x, y, t) \phi(\xi) = \exp \left\{ i \left( \sigma \cdot t + \sum_{k=1}^{n} \left( x_k \xi_k + \frac{1}{2} x_k y_k \right) (\sigma \cdot A_k) \right) \right\} \phi(\xi + y), \tag{3.9} \]

for \( \phi \in L^2(\mathbb{R}^n) \) and \( \xi \in \mathbb{R}^n \). The operators \( \pi^A_{\sigma}(x, y, t) \) are bounded in \( L^2(\mathbb{R}^n) \) for each \( (x, y, t) \in G_A \), and one can check that they define unitary representations on \( G_A \) and irreducible if and only if \( \sigma \cdot A_k \neq 0 \) for each \( k \).
The group Fourier transform is defined, for each $f \in L^1(G_A)$, as the operator
\[ \pi^A_\sigma(f) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \]
given by
\[ \pi^A_\sigma(f)(\phi) = \int_{G_A} f(x, y, t) \pi^A_\sigma(x, y, t) \phi \, dx \, dy \, dt. \] (3.10)
\[ \pi^A_\sigma(f) \] is a bounded operator on $L^2(\mathbb{R}^n)$, and we have the following Plancherel theorem.

**Theorem 3.1** (Plancherel theorem for $G_A$). If $f \in L^1(G_A) \cap L^2(G_A)$ then
\[ ||f||^2_{L^2(G_A)} = \int_{\mathbb{R}^d} \text{tr} \left( (\pi^A_\sigma(f))^* \pi^A_\sigma(f) \right) D(\sigma) \, d\sigma, \] (3.11)
where $D(\sigma)$ is given by
\[ D(\sigma) = (2\pi)^{-(n+1)} \prod_{k=1}^n |\sigma \cdot A_k|. \] (3.12)

The proof of this theorem is similar to the proof in the Heisenberg group case (see for example [Tha98]) and can be found in [NRS01]. Here $\text{tr}(T)$ is the trace of the operator $T$, and thus $\text{tr}(T^*T)$ is the square of its Hilbert-Schmidt norm. Polarizing (3.11) we obtain
\[ \int_{G_A} f(z, t) g(z, t) \, dz \, dt = \int_{\mathbb{R}^d} \text{tr} \left( (\pi^A_\sigma(g))^* \pi^A_\sigma(f) \right) D(\sigma) \, d\sigma, \]
and thus we have the following inversion formula.

**Corollary 3.2** (Inversion formula). For appropriate functions $f$ we have the inversion formula
\[ f(z, t) = \int_{\mathbb{R}^d} \text{tr} \left( (\pi^A_\sigma(z, t))^* \pi^A_\sigma(f) \right) D(\sigma) \, d\sigma. \] (3.13)
As “appropriate functions” we can take, for instance, functions in the Schwartz class $\mathcal{S}(G_A)$ [FS82].
Examples

1. In the Heisenberg group we have the inversion formula

\[ f(z, t) = (2\pi)^-(n+1) \int_{\mathbb{R}} \text{tr} \left( \pi^A_\sigma(z, t) \pi^A_\sigma(f) \right) |\sigma|^n d\sigma. \]

Note that here \( A = (1 \ldots 1) \) and \( d = 1 \).

2. The inversion formula in \( (\mathbb{H}_1)^n \) is given by

\[ f(z, t) = (2\pi)^-(n+1) \int_{\mathbb{R}^n} \text{tr} \left( \pi^A_\sigma(z, t) \pi^A_\sigma(f) \right) \prod_{k=1}^n |\sigma_k| d\sigma, \]

which is a “tensor product” of the corresponding formula for \( \mathbb{H}_1 \) (since the representations of \( (\mathbb{H}_1)^n \) are given in this way).

Recall that \( G_A \) is isomorphic to the quotient \( (\mathbb{H}_1)^n/K \) where \( K \) is the kernel of the map \( I \otimes A \). Here \( A \) is denoting the linear map \( x \mapsto Ax \) from \( \mathbb{R}^n \) onto \( \mathbb{R}^d \). The transpose of this map is given by the map \( A^T : \mathbb{R}^d \to \mathbb{R}^n \) given by

\[ \sigma \mapsto (\sigma \cdot A_1, \ldots, \sigma \cdot A_n). \]

The range of \( A^T \) is the orthogonal complement \( K^\perp \) of \( K \) in \( \mathbb{R}^n \) \( (K \cong \ker(A)) \), and if \( \lambda = A^T \sigma \) then

\[ \pi^I_\lambda(z, t) = \pi^A_\sigma(z, At), \]

where \( \pi^I_\lambda(z, t) \) is the tensor product

\[ \pi_{\lambda_1}(z_1, t_1) \otimes \ldots \otimes \pi_{\lambda_n}(z_n, t_n) \]

of representations of \( \mathbb{H}_1 \). As noted above, \( \pi^A_\sigma \) is irreducible if and only if all products \( \sigma \cdot A_k \) are nonzero, i.e. each coordinate \( \lambda_k \) is not zero. Thus every irreducible unitary representation of \( G_A \) lifts to an irreducible unitary representation of \( (\mathbb{H}_1)^n \).
Conversely, an irreducible representation $\pi_l^n$ projects to $G_A$ if and only if $\pi_l^n$ is trivial on $K$, if and only if $\pi_l^n(0, t) = 1$ for all $t \in K$, if and only if $e^{i\lambda t} = 1$ for all $t \in K$, if and only if $\lambda \in K^\perp$.

The representations $\pi^A_\sigma$ induce a mapping from the differential operators on $G_A$ to operators acting on functions on $\mathbb{R}^n$. Indeed, if $X$ is a vector field on $G_A$, we define

$$\pi^A_\sigma(X)\varphi = \frac{d}{ds}\pi^A_\sigma(\exp sX)\varphi\bigg|_{s=0},$$

and we extend to differential operators in the obvious way. Thus the mappings $\pi^A_\sigma$ send the basis elements $X_k, Y_k, T_j$ to the operators

$$\begin{align*}
\pi^A_\sigma(X_k)\varphi(\xi) &= i(\sigma \cdot A_k)\xi_k\varphi(\xi) \\
\pi^A_\sigma(Y_k)\varphi(\xi) &= \frac{\partial\varphi}{\partial\xi_k} \\
\pi^A_\sigma(T_j)\varphi(\xi) &= i\sigma_j\varphi(\xi),
\end{align*}$$

and in particular the sublaplacian $\mathcal{L}$ is sent to the operator

$$\pi^A_\sigma(\mathcal{L})\varphi(\xi) = \left(-\Delta + \sum_{k=1}^n |\sigma \cdot A_k|^2\xi_k^2\right)\varphi(\xi).$$

The operator is a “weighted” version of the harmonic oscillator operator, defined, on $C^2(\mathbb{R})$, by

$$H_\mu = -\frac{d^2}{dt^2} + \mu^2 t^2, \quad \mu > 0.$$  

The eigenfunctions of $H_\mu$ are given by the renormalized Hermite functions $h^\mu_k(t) = \mu^{1/4}h_k(\sqrt{\mu}t)$, where

$$h_k(t) = \left(2^k \sqrt{\pi} k!\right)^{-1/2} e^{t^2/2} \frac{d^k}{dt^k} e^{-t^2}.$$
The eigenvalue that correspond to $h^\mu_k$ is $\mu(2k + 1)$. Hence, if we define the functions

$$h^{\sigma,A}_\alpha(t_1, \ldots, t_n) = \prod_{k=1}^n h^{|\sigma,A_k|}(t_k),$$

then $h^{\sigma,A}_\alpha$ is an eigenfunction of (3.15) with eigenvalue

$$\sum_{k=1}^n |\sigma \cdot A_k|(2\alpha_k + 1).$$

Moreover, the functions $h^\mu_k$ form a complete orthonormal system in $L^2(\mathbb{R})$, so the functions $h^{\sigma,A}_\alpha$ form a complete orthonormal system in $L^2(\mathbb{R}^n)$, and thus

$$L^2(\mathbb{R}^n) = \bigoplus_{\alpha \in \mathbb{N}^n} V_\alpha,$$

where $V_\alpha = \text{span}\{h^{\sigma,A}_\alpha\} \subset L^2(\mathbb{R}^n)$ is the spectral decomposition for the operator (3.15).

One can use the decomposition (3.17) to obtain a decomposition for the sublaplacian in $L^2(G_A)$. Indeed, the eigenfunctions of $\mathcal{L}$ are given by the functions

$$\phi^{\sigma,A}_{\alpha,\beta}(z, t) = \langle \pi^A_{\sigma}(z, t) h^{\sigma,A}_\alpha, h^{\sigma,A}_\beta \rangle,$$

where $\langle , \rangle$ denotes standard inner product in $L^2(\mathbb{R}^n)$. The functions (3.18) form a complete orthonormal system in $L^2(G_A)$. This is proven in the same way as in the Heisenberg group, and the proof in that case can be found, for instance, in [Tha98]. Hence we have the following theorem.

**Theorem 3.3.** For an appropriate function $f$ (say in the Schwartz class $\mathcal{S}(G_A)$) we have the expansion

$$f(z, t) = \sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}^d} f \ast \overline{\phi^{\sigma,A}_{\alpha,\alpha}}(z, t) D(\sigma) d\sigma,$$
where $D(\sigma)$ is the Plancherel measure given by (3.12).

**Proof.** From the inversion formula (3.13)

$$f(z,t) = \int_{\mathbb{R}^d} \text{tr} \left( \pi^A(\sigma) f \right) D(\sigma) d\sigma$$

$$= \int_{\mathbb{R}^d} \sum_{\alpha} \langle \pi^A(\sigma) f, h^{\sigma \cdot A}_{\alpha} \rangle D(\sigma) d\sigma,$$

where we have calculated the trace of $\pi^A(\sigma) f$ as

$$\sum_{\alpha} \langle \pi^A(\sigma) f, h^{\sigma \cdot A}_{\alpha} \rangle,$$

since the functions $h^{\sigma \cdot A}_{\alpha}$ form a complete orthonormal system in $L^2(\mathbb{R}^n)$ (where the operator $\pi^A(\sigma) f$ acts). Now

$$\langle \pi^A(\sigma) f, h^{\sigma \cdot A}_{\alpha} \rangle = \langle \pi^A(f), \pi^A(\sigma) h^{\sigma \cdot A}_{\alpha} \rangle$$

$$= \int_{G_A} f(w,s) \overline{\phi^{\sigma \cdot A}_{\alpha}(w,s)^{-1}(z,t)} dw ds$$

$$= \int_{G_A} f(w,s) \overline{\phi^{\sigma \cdot A}_{\alpha}(w,s)^{-1}(z,t)} dw ds$$

$$= f * \phi^{\sigma \cdot A}_{\alpha}(z,t),$$

and the theorem follows. \qed

One can explicitly calculate the functions $\varphi^{\sigma \cdot A}_{\alpha} = \overline{\phi^{\sigma \cdot A}_{\alpha}}$ in terms of Laguerre polynomials. Indeed, we have

$$\varphi^{\sigma \cdot A}_{\alpha}(z,t) = e^{-i\sigma \cdot t} \prod_{k=1}^{n} L_{\alpha_k} \left( \frac{1}{2} |\sigma \cdot A_k||z_k|^2 \right) e^{-\frac{1}{2} |\sigma \cdot A_k||z_k|^2}, \quad (3.20)$$

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where the $L_k$ are the Laguerre polynomials given by

$$L_k(t) = \frac{e^t}{k!} \frac{d^k}{dt^k} (e^t e^{-t}).$$

See [Tha98] for a proof of this result.

We say that a function $f(z,t)$ on $G_A$ is polyradial if $f$ depends only on the quantities $|z_1|, |z_2|, \ldots, |z_n|$ and $t$. Let $\mathcal{A}$ be the subalgebra of $L^1(G_A)$ of polyradial functions. The Gelfand transform\(^1\) of a function $f \in \mathcal{A}$ is given by

$$\hat{f}(\alpha, \sigma) = \langle \pi_{\sigma}^A(f) h_{\alpha}^{\sigma, A}, h_{\alpha}^{\sigma, A} \rangle = \int_{G_A} f(z,t) \phi_{\alpha, \sigma}^{\sigma, A}(z,t) \, dz \, dt, \quad \alpha \in \mathbb{N}^n, \sigma \in \mathbb{R}^d. \quad (3.21)$$

We can also obtain a Plancherel formula, and thus an inversion formula, for the Gelfand transform (3.21). One first observes that if $f$ is polyradial, then for each $\alpha \in \mathbb{N}^n, \sigma \in \mathbb{R}^d$ with $\sigma \cdot A_k \neq 0$ for every $k$ the functions $h_{\alpha}^{\sigma, A}$ are eigenfunctions of the operator $\pi_{\sigma}^A(f)$. This follows from the property

$$\phi_{\alpha, \beta}^{\sigma, A}(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n, t) = e^{\sum (\beta_k - \alpha_k) \theta_k} \phi_{\alpha, \beta}^{\sigma, A}(z, t), \quad (3.22)$$

and the fact that $h_{\alpha}^{\sigma, A}$ is an eigenfunction of $\pi_{\sigma}^A(f)$ if and only if

$$\langle \pi_{\sigma}^A(f) h_{\alpha}^{\sigma, A}, h_{\beta}^{\sigma, A} \rangle = 0$$

whenever $\alpha \neq \beta$. A proof of (3.22) can be found in [Tha93]. Thus

$$\pi_{\sigma}^A(f) h_{\alpha}^{\sigma, A} = \hat{f}(\alpha, \sigma) h_{\alpha}^{\sigma, A},$$

\(^1\)The maximal ideal of $\mathcal{A}$ also contains points other than the pairs $(\alpha, \sigma)$, but these are irrelevant for our discussion since they have zero Plancherel measure.
by (3.21), and we have, for $f$ and $g$ appropriate functions in $A$,

$$
\int_{G_A} f(z,t)g(z,t)dzdt = \int_{\mathbb{R}^d} \text{tr} \left( \pi^A_{\sigma}(g)^* \pi^A_{\sigma}(f) \right) D(\sigma)d\sigma
$$

$$
= \int_{\mathbb{R}^d} \sum_{\alpha \in \mathbb{N}^n} \langle \pi^A_{\sigma}(g)^* \pi^A_{\sigma}(f) h^{\sigma,A}_{\alpha}, h^{\sigma,A}_{\alpha} \rangle D(\sigma)d\sigma
$$

$$
= \sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}^d} \langle \pi^A_{\sigma}(f) h^{\sigma,A}_{\alpha}, \pi^A_{\sigma}(g) h^{\sigma,A}_{\alpha} \rangle D(\sigma)d\sigma
$$

$$
= \sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}^d} \hat{f}(\alpha,\sigma) \overline{\hat{g}(\alpha,\sigma)} D(\sigma)d\sigma.
$$

Therefore we have the Plancherel formula

$$
\int_{G_A} |f(z,t)|^2 dzdt = \sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}^d} |\hat{f}(\alpha,\sigma)|^2 D(\sigma)d\sigma, \quad (3.23)
$$

and the inversion formula

$$
f(z,t) = \int_{\mathbb{R}^m} \text{tr} \left( \pi^A(z,t)^* \pi^A_{\sigma}(f) \right) D(\sigma)d\sigma
$$

$$
= \sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}^d} \langle \pi^A(z,t) h^{\sigma,A}_{\alpha}, \pi^A_{\sigma}(f) h^{\sigma,A}_{\alpha} \rangle D(\sigma)d\sigma
$$

$$
= \sum_{\alpha \in \mathbb{N}^n} \int_{\mathbb{R}^d} \hat{f}(\alpha,\sigma) \varphi^{\sigma,A}_{\alpha}(z,t) D(\sigma)d\sigma. \quad (3.24)
$$

### 3.2 Projection operators

Let $G$ be a nilpotent group and $T$ the left-translation invariant operator with convolution kernel the tempered distribution $K$. For a normal subgroup $N$ of $G$, we define the operator $T_N$, on functions $f \in S(G/N)$, by

$$
T_N f(x) = K(f_x), \quad x \in G/N, \quad (3.25)
$$
where $f_x$ is the function on $G$ whose value at $y \in G$ is given by

$$f_x(y) = f(x\pi(y)^{-1}),$$

where $\pi : G \to G/N$ is the fundamental projection mapping. In the case when the distribution $K$ is a function in $L^1(G)$, the operator $T_N$ is given by\(^2\)

$$T_N f(x) = \int_G f(x\pi(y)^{-1})K(y)dy, \quad x \in G/N.$$  \hfill (3.26)

In such case, the convolution kernel $K_N$ of $T_N$ is given by, for a left-coset $xN \in G/N$,

$$K_N(xN) = \int_N K(xy)dy.$$  \hfill (3.27)

**Proposition 3.4.** Assume that $K \in L^1(G)$ and that the operator $Tf \mapsto f \ast K$ is bounded on $L^p(G)$. Then, the operator $T_N$ extends to a bounded operator on $L^p(G/N)$ and

$$||T_N||_{L^p(G/N) \to L^p(G/N)} \leq ||T||_{L^p(G) \to L^p(G)}.$$

This proposition is a special case of Theorem 2.4 in [CW76], and we refer the reader to that monograph for its proof.

In the situation that concerns us, $G = (\mathbb{H}_1)^n$, and $N = \ker(I \otimes A) \cong \mathbb{R}^{n-d}$, and $G/N \cong G_A$. Let $A^*$ be the transpose of $A$ and $B$ an $n \times (n-d)$-matrix that maps $\mathbb{R}^{(n-d)}$ onto $\ker A$. Thus, the convolution kernel $K_N$ is given by

$$K_N(z,t) = \int_{\mathbb{R}^{n-d}} K(z,A^*(AA^*)^{-1}t + Bs)ds, \quad (z,t) \in G_A.$$  

\(^2\)A will denote Haar measure, either in $G$, $G/N$ or $N$, depending on the context.
The Gelfand transform $\kappa_N$ of $K_N$ is then given by

$$\kappa_N(\alpha, \sigma) = \kappa(\alpha, A^*\sigma), \quad \alpha \in \mathbb{N}^n, \sigma \in (\mathbb{R} \setminus \{0\})^d,$$

where $\kappa$ is the Gelfand transform of $K$. To show (3.28), observe that

$$A^*\sigma = \begin{pmatrix} \sigma \cdot A_1 \\ \vdots \\ \sigma \cdot A_n \end{pmatrix}.$$ 

Thus

$$\kappa_N(\alpha, \sigma) = \int_{G_A} K_N(z, t) e^{it\cdot\sigma} \prod_{k=1}^n L_{\alpha_k} \left( \frac{1}{2} |\sigma \cdot A_k||z_k|^2 \right) e^{-\frac{1}{4} |\sigma \cdot A_k||z_k|^2} dzdt$$

$$= \int_{\mathbb{C}^n \times \mathbb{R}^d} \int_{\mathbb{R}^{n-d}} K(z, A^*(AA^*)^{-1}t + Bs) e^{it\cdot\sigma}$$

$$\times \prod_{k=1}^n L_{\alpha_k} \left( \frac{1}{2} |\sigma \cdot A_k||z_k|^2 \right) e^{-\frac{1}{4} |\sigma \cdot A_k||z_k|^2} dsdzdt$$

$$= \int_{\mathbb{C}^n} \int_{\mathbb{R}^d} K(z, u) e^{iu\cdot\sigma} \prod_{k=1}^n L_{\alpha_k} \left( \frac{1}{2} |\sigma \cdot A_k||z_k|^2 \right) e^{-\frac{1}{4} |\sigma \cdot A_k||z_k|^2} dzdt$$

$$= \kappa(\alpha, A^*\sigma),$$

where we have made the change of variables $u \mapsto A^*(AA^*)^{-1}t + Bs$.

### 3.3 Marcinkiewicz multipliers on $G_A$

In this section we introduce the study of the boundedness of the operator

$$T = m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT_1, \ldots, -iT_d),$$
on the group $G_A = (H_1)^n / \ker A$, provided sufficient regularity conditions on the function $m$ on $\mathbb{R}^n \times \mathbb{R}^d$. The technique will be one of “lifting”, i.e. we will study the boundedness of the operator (3.31) on the quotient by first lifting it to a proper operator $\mathcal{M}$ on the product group $(H_1)^n$, and then, through the transference method of the previous section, deduce the boundedness of (3.31) from the boundedness of $\mathcal{M}$. Such method of lifting has been using, for example, in the study of averages over non-homogeneous curves, where the lifting technique provides a way to use homogeneity results. See Chapter IX of [Ste93], for example, and references therein.

The remarks of the previous section suggest that we lift the operator (3.31) to the operator

$$E m(L_1^\#, \ldots, L_n^\#, -iT_1^\#, \ldots, -iT_n^#),$$

where the operators $L_k^\#$ and $T_k^\#$ act on each coordinate of the product $(H_1)^n$, and the function $E m$ is defined as

$$E m(x, \lambda) = m(x, (AA^*)^{-1} A\lambda).$$

The operator (3.29) is now a multiplier operator on $(H_1)^n$, so we will be able to use the results of Chapter 2, provided we have the conditions on $m$ that will guarantee that $E m$ satisfies the conditions of Corollary 2.5.

We start with the following result, straightforward consequence of Proposition (3.4).

**Proposition 3.5.** Let $m$ be a bounded continuous function on $\mathbb{R}_+^n \times \mathbb{R}^n$ such that the operator

$$\mathcal{M} = m(L_1^\#, \ldots, L_n^\#, -iT_1^\#, \ldots, -iT_n^#)$$

is bounded in $L^p(G)$ and has its kernel in $L^1(G)$. Then, the operator given by

$$T = \tilde{m}(L_1, \ldots, L_n, -iT_1, \ldots, -iT_d),$$

is
where \( \tilde{m}(x, y) = m(x, A^*y) \) for \( x \in \mathbb{R}^n_+ \) and \( y \in \mathbb{R}^d \), is bounded in \( L^p(G_A) \).

As above, the operators \( L^\#_k \) and \( T^\#_k \) act on each coordinate of the product \((\mathbb{H}_1)^n\).

**Proof.** This proposition follows from Proposition 3.4 and the fact that the Gelfand transforms of the kernels of \( M \) and \( T \) are \( m(\alpha, \lambda) \) and \( \tilde{m}(\alpha, \sigma) \), respectively, by equation (3.28). \( \square \)

**Theorem 3.6.** Let \( m \) be a bounded smooth function on \( \mathbb{R}^n_+ \times \mathbb{R}^n \) and suppose that, for some \( s_k > 1, s'_k > 1/2 \), \( m \in \mathcal{L}_{s,s',\text{loc}}(\mathbb{R}^n_+ \times \mathbb{R}^n) \). Define \( \tilde{m} \) on \( \mathbb{R}^n_+ \times \mathbb{R}^d \) by \( \tilde{m}(x, y) = m(x, A^*y) \). Then the operator \( T \) given by (3.31) is bounded in \( L^p(G_A) \), \( 1 < p < \infty \), with norm

\[
||T||_{L^p \to L^p} \lesssim ||m||_{\mathcal{L}_{s,s',\text{loc}}}.
\]

**Proof.** Let \( \phi_0 \in C_0^\infty(\mathbb{R}) \), supported in \((1/2, 2)\), be such that

\[
\sum_{j \in \mathbb{Z}} \phi(2^{-j}|x|) = 1
\]

for all \( x \neq 0 \). Define the functions \( \phi_j \) and \( \tilde{\phi}_l \), for \( j \in \mathbb{Z} \) and \( l = (l_1, \ldots, l_n) \in \mathbb{Z}^n \), by

\[
\phi_j(x) = \phi(2^{-j}x), \quad x \in \mathbb{R}_+,
\]

\[
\tilde{\phi}_l(y) = \phi(2^{-l_1}|y_1|) \cdots \phi(2^{-l_n}|y_n|), \quad y \in \mathbb{R}^n,
\]

and

\[
\Phi_{j,l}(x, y) = \begin{cases} 
\phi_j(x_1 + \ldots + x_n)\tilde{\phi}_l(y) & x_1, \ldots, x_n > 0 \\
0 & \text{otherwise.}
\end{cases}
\]

We thus define, for \( j \in \mathbb{Z} \) and \( l = (l_1, \ldots, l_n) \in \mathbb{Z}^n \), the functions

\[
m_{j,l}(x, y) = m(x, y)\Phi_{j,l}(x, y).
\]
Then we have $\sum_{j,l} m_{j,l}(x, y) = m(x, y)$ for every $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$, and, moreover, the series

$$\sum_{j,l} m_{j,l} = m \quad \text{in } L_{s,s'}_{loc}. \quad (3.32)$$

To see (3.32), note that

$$\sup_{r \in \mathbb{R}^n_+} ||m_{j,l}^r \eta||_{L^2(L^2_{r})} \leq ||m||_{L_{s,s'}_{loc}},$$

and moreover, if $J \in \mathbb{Z}_+, L = (L_1, \ldots, L_n) \in \mathbb{Z}^n_+$, and

$$m_{J,L} = \sum_{|j| \leq J} \sum_{|l| \leq L_k} m_{j,l},$$

then

$$\sup_{r \in \mathbb{R}^n_+} ||m_{J,L}^r \eta||_{L^2(L^2_{r})} \lesssim ||m||_{L_{s,s'}_{loc}},$$

as only a fixed finite number of $m_{j,l}^r$ in the sum $m_{J,L}^r$ are nonzero for each $r > 0$. Thus $m_{J,L} \to m$ in $L_{s,s'}_{loc}$.

By Theorem 2.5, the operators

$$\mathcal{M}_{J,L} = m_{J,L}(\mathcal{L}_{1}^\#, \ldots, \mathcal{L}_{n}^\#, -iT_{1}^\#, \ldots, -iT_{n}^\#)$$

are bounded on $L^p(G)$, with bounds uniformly estimated by $||m||_{L_{s,s'}_{loc}}$. Thus, for $f \in L^p(G)$, $\lim \mathcal{M}_{J,L}f = \mathcal{M}f$ in $L^p(G)$.

Moreover, as each $m_{J,L}$ is a $C^\infty$ function of compact support, each $\mathcal{M}_{J,L}$ has kernel in $L^1(G)$. Then the operators

$$\mathcal{T}_{J,L} = \tilde{m}_{J,L}(\mathcal{L}_{1}, \ldots, \mathcal{L}_{n}, -iT_{1}, \ldots, -iT_{d}),$$

where $\tilde{m}_{J,L}(x, y) = m_{J,L}(x, A^*y)$ for $x \in \mathbb{R}^n_+ \times \mathbb{R}^d$, are uniformly bounded on $L^p(G_A)$. 
with norms
\[ \| T_{J,L} \|_{L^p \to L^p} \lesssim \| m \|_{L^{s,s',\text{loc}}} \]
uniformly in \( J \) and \( L \).

Therefore \( T = \lim T_{J,L} \) is bounded in \( L^p(G_A) \), with norm
\[ \| T \|_{L^p \to L^p} \lesssim \| m \|_{L^{s,s',\text{loc}}} . \]

We have then the following result, straightforward consequence of Theorem 3.6.

**Corollary 3.7.** Let \( m \) be a bounded continuous function on \( \mathbb{R}^n_+ \times \mathbb{R}^d \) and suppose that, for some \( s_k > 1, s'_k > 1/2, k = 1, \ldots, n \), we have that
\[
\sup_{r \in \mathbb{R}^n_+} \int_{\mathbb{R}^{n} \times \mathbb{R}^{n}} \prod_{k=1}^{n} \left( 1 + |2\pi \xi_k|^2 \right)^{s_k} \left( 1 + |2\pi \xi_k| + |2\pi \sigma| \right)^{2s'_k} \left| \mathcal{E}m \right|^{r} \eta(\xi,\sigma) \right|^2 d\xi d\sigma < \infty,
\]
(3.33)
where \( \eta \) is the function defined in (2.2). Then the operator \( T \) of (3.31) is bounded in \( L^p(G_A) \), with norm uniformly estimated by (3.33).

This theorem implies that a regularity condition of order
\[ s_1 + \ldots + s_n + s'_1 + \ldots + s'_n > 3n/2 \]
is sufficient for the boundedness of the operator (3.31) on \( L^p(G_A) \).

As an special example of a Marcinkiewicz multiplier, we consider the operator \( m(\mathcal{L}) \), where \( m \) is a bounded function on \( \mathbb{R} \), and \( \mathcal{L} \) is the sublaplacian on \( G_A \) given by (3.8).

Note that, in this case, the function \( \mathcal{E}m \) defined in (3.30) now takes the form \( \mathcal{E}m(x,\lambda) = m(x_1 + \ldots + x_n) \), and thus we have the following corollary, analogous to Corollary 2.6.
Corollary 3.8. Let $m$ be a bounded continuous function on $\mathbb{R}$, such that, for some $s_0 > 3n/2$, $m \in L^2_{s_0,\text{loc}}(\mathbb{R})$. Then $m$ is a multiplier, i.e. the operator $m(\mathcal{L})$ is bounded in $L^p(G_A)$. 
Chapter 4

Products of $H$-type groups

4.1 $H$-type groups

Let $\mathfrak{g}$ be a finite dimensional 2-step nilpotent Lie algebra equipped with an inner product $(\cdot, \cdot)$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$ and $\mathfrak{w}$ its orthogonal complement with respect to $(\cdot, \cdot)$. Hence $\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{z}$, this decomposition is orthogonal and $[\mathfrak{w}, \mathfrak{w}] \subset \mathfrak{z}$.

We say that $\mathfrak{g}$ is an $H$-type algebra if, for every unitary $Z \in \mathfrak{z}$, the map $J_Z : \mathfrak{w} \to \mathfrak{w}$ that satisfies

$$(JZW_1, W_2) = (Z, [W_1, W_2])$$

is orthogonal. Since $J_Z^2 = -J_Z$, this is equivalent to $J_Z^2 = -|Z|^2$ for every $Z \in \mathfrak{z}$. Moreover, it is also equivalent to the fact that the mapping $\text{ad}^*_W : \mathfrak{z}^* \to \mathfrak{w}^*$ is an isometry for every unitary $W \in \mathfrak{w}$.

The above imply that $\dim \mathfrak{w}$ is even, say $\dim \mathfrak{w} = 2n$, $[\mathfrak{w}, \mathfrak{w}] = \mathfrak{z}$ and, if $d = \dim \mathfrak{z}$, $d \leq 2n$.

We say that the connected, simply connected Lie group $G$ is an $H$-type group if its Lie algebra $\mathfrak{g}$ is an $H$-type algebra. Since $G$ is nilpotent, $G = \exp \mathfrak{g}$, and we write
the elements of $G$ as $(w, z)$, where $w \in \exp \mathfrak{w}$ and $z \in \exp \mathfrak{z}$. Then

$$(w, z) = \exp(W + Z) = \exp W \cdot \exp Z,$$

if $w = \exp W$ and $z = \exp Z$. We denote the identity in $G$ as $(0, 0)$.

If we denote $\exp \frac{1}{2}[W_1, W_2]$ by $\sigma(\exp W_1, \exp W_2)$, by the Campbell-Hausdorff formula we can write the operation in $G$ as

$$(w_1, z_2) \cdot (w_2, z_2) = (w_1 + w_2, z_1 + z_2 + \sigma(w_1, w_2)), \quad (4.1)$$

as $\mathfrak{g}$ is 2-step nilpotent.

An $H$-type group $G$ is a homogeneous group with dilations

$$\delta_r(\exp W, \exp Z) = (\exp rW, \exp r^2Z),$$

which are automorphisms in the group by (4.1). We will denote $\delta_r(w, z)$ as

$$r \cdot (w, z) = (r \cdot w, r^2 \cdot z).$$

A homogenous norm is given by

$$|(\exp W, \exp Z)| = (|W|^4 + |Z|^2)^{1/4},$$

where $|W|$ and $|Z|$ denote the Euclidean norms induced by the inner product on $\mathfrak{w}$ and $\mathfrak{z}$, respectively. We will denote $|(w, 0)|$ and $|(0, z)|$ simply by $|w|$ and $|z|$, respectively.

We say that a function $f$ on $G$ is $\mathfrak{w}$-radial if $f(w, z)$ only depends on $|w|$. If $\mathcal{A}$ denotes the closed subspace of $L^1(G)$ formed by the $\mathfrak{w}$-radial integrable functions on $G$, then $\mathcal{A}$ is a commutative, semisimple subalgebra of $L^1(G)$ whose Gelfand spectrum
can be parametrized by
\[ \mathbb{R}_+ \cup \mathbb{N} \times (\mathfrak{z}^* \setminus \{0\}) \],
where \( \mathfrak{z}^* \) is the dual space of \( \mathfrak{z} \). (see [MRS96], [YZ08], and references therein). The Gelfand transform of \( f \in \mathcal{A} \) is then given by
\[ \hat{f}(\pi) = \int_G f(w, z) \phi_{\pi}(w, z) dwdz, \]
where, if \( \pi \in \mathbb{R}_+ \),
\[ \phi_{\pi}(w, z) = e^{i\pi \langle w, w_0 \rangle}, \]
for a fixed \( w_0 \in W \), and, if \( \pi = (\alpha, \lambda) \in \mathbb{N} \times (\mathfrak{z}^* \setminus \{0\}) \),
\[ \phi_{\alpha,\lambda}(w, z) = \left( \frac{\alpha + n - 1}{\alpha} \right)^{-1} e^{-i\lambda(z)} e^{-\frac{||w||^2}{4}} L_{\alpha}^{n-1} \left( \frac{1}{2} ||\lambda||^2 \right). \] (4.2)
Note that we define \( \langle \cdot, \cdot \rangle \) in \( G \) through the exponential mapping from \( \mathfrak{g} \), i.e.
\[ \langle g, h \rangle = (\exp^{-1} g, \exp^{-1} h), \]
where the right side is just the inner product on \( \mathfrak{g} \).

One can prove the Plancherel formula for \( f \in \mathcal{A} \cap G \)
\[ \|f\|_{L^2(G)}^2 = C_{\mathfrak{g}} \int_{\mathfrak{z}^* \setminus \{0\}} \sum_{\alpha=0}^{\infty} \left| \hat{f}(\alpha, \lambda) \right|^2 \left( \frac{\alpha + n - 1}{\alpha} \right) |\lambda|^n d\lambda, \] (4.3)
where \( d\lambda \) is the Lebesgue measure on the vector space \( \mathfrak{z}^* \) and the constant \( C_{\mathfrak{g}} \) depends only on \( n \) and \( d \). Polarization of the Plancherel formula (4.3) gives the inversion formula
\[ f(w, z) = C_{\mathfrak{g}} \int_{\mathfrak{z}^* \setminus \{0\}} \sum_{\alpha=0}^{\infty} \hat{f}(\alpha, \lambda) \phi_{\alpha,\lambda}(w, z) \left( \frac{\alpha + n - 1}{\alpha} \right) |\lambda|^n d\lambda, \] (4.4)
convergent, say, for \( f \in \mathcal{S}(G) \), the Schwartz space of \( G \).
Fix an orthonormal basis $W_1, \ldots, W_{2n}$ of $\mathfrak{w}$. If we define the sub-Laplacian

$$\mathcal{L} = -\sum_{j=1}^{2n} W_j^2,$$

then $\mathcal{L}$ maps $\mathcal{A} \cap C^\infty(G)$ into itself, and the spectral properties of the Laguerre polynomials show that

$$\hat{\mathcal{L}}f(\alpha, \lambda) = (2\alpha + n)|\lambda|\hat{f}(\alpha, \lambda)$$

and, for $Z \in \mathfrak{z}$,

$$\hat{Z}f(\alpha, \lambda) = i\lambda(Z)\hat{f}(\alpha, \lambda).$$

(4.5)

(4.6)

4.2 Products of $H$-type groups

Consider now the product $G = G_1 \times \cdots \times G_N$ of $N$ $H$-type groups $G_j$, each of dimension $2n_j + d_j$. Set $n = n_1 + \ldots + n_N$ and $d = d_1 + \ldots + d_N$, so that $G$ has dimension $2n + d$.

The Lie algebra of $G$ is then the direct sum of the $N$ $H$-type algebras $\mathfrak{g}_j$ of each group $G_j$, respectively, and we set

$$\mathfrak{w} = \mathfrak{w}_1 \oplus \cdots \oplus \mathfrak{w}_N \quad \text{and} \quad \mathfrak{z} = \mathfrak{z}_1 \oplus \cdots \oplus \mathfrak{z}_N.$$

As before, we write each element in $G$ as $(w, z)$, where $w \in \exp \mathfrak{w}$ and $z \in \exp \mathfrak{z}$, and we note that

$$w = (w_1, \ldots, w_N) \quad \text{and} \quad z = (z_1, \ldots, z_N),$$

where each $w_j \in \exp \mathfrak{w}_j$ and $z_j \in \exp \mathfrak{z}_j$. Also, $G$ is a homogeneous group with dilations

$$\delta_r(\exp W, \exp Z) = (\exp rW, \exp r^2Z),$$

and a homogeneous norm is given by the Euclidean norm induced by the homogenous
norm on each $G_j$.

We say that a function $f(w, z)$ on $G$ is \textit{polyradial} if it depends only on $|w_1|, \ldots, |w_N|$. We write the subalgebra of $L^1(G)$ of polyradial functions as $\mathcal{A}$. The Gelfand spectrum is then parametrized by

$$
\prod_{j=1}^N \left( \mathbb{R}_+ \cup \mathbb{N} \times (\mathfrak{z}_j^* \setminus \{0\}) \right),
$$

and the Gelfand transform on $\mathcal{A}$ is then given by, for

$$
(\alpha, \lambda) = (\alpha_1, \ldots, \alpha_N, \lambda_1, \ldots, \lambda_N) \in \prod_{j=1}^N \left( \mathbb{N} \times (\mathfrak{z}_j^* \setminus \{0\}) \right),
$$

$$
\hat{f}(\alpha, \lambda) = \int_G f(w, z) \phi^G_{\alpha, \lambda}(w, z) dwdz, \quad (4.7)
$$

where the functions $\phi^G_{\alpha, \lambda}$ are the tensor products of the respective $\phi$ functions on each group $G_j$,

$$
\phi^G_{\alpha, \lambda}(w, z) = \prod_{j=1}^N \phi^{G_j}_{\alpha_j, \lambda_j}(w, z), \quad (4.8)
$$

and $\phi^{G_j}_{\alpha_j, \lambda_j}(w, z)$ is given by (4.2). The Plancherel formula is then given by

$$
\|f\|_{L^2(G)}^2 = C_G \int_{\prod_{j=1}^N \mathfrak{z}_j^* \setminus \{0\}} \sum_{\alpha \in \mathbb{N}^N} |\hat{f}(\alpha, \lambda)|^2 D_G(\lambda) d\lambda_1 \ldots d\lambda_N, \quad (4.9)
$$

where

$$
D_G(\lambda) = \prod_{j=1}^N \left( \frac{\alpha_j + n_j - 1}{\alpha_j} \right) |\lambda_j|^{n_j},
$$

and, by polarization, we have the respective inversion formula for, say, $f \in \mathcal{S}(G)$,

$$
f(w, z) = C_G \int_{\prod_{j=1}^N \mathfrak{z}_j^* \setminus \{0\}} \sum_{\alpha \in \mathbb{N}^N} \hat{f}(\alpha, \lambda) \overline{\phi^G_{\alpha, \lambda}(w, z)} D_G(\lambda) d\lambda_1 \ldots d\lambda_N. \quad (4.10)
$$

Fix orthonormal bases $W^{1}_{j}, \ldots, W^{n_{j}}_{j}$ of each $w_{j}$. We define the sub-Laplacian on
\( \mathbf{w} \) as

\[
\mathcal{L} = - \sum_{1 \leq i \leq n_j} \sum_{1 \leq j \leq N} (W^i_j)^2,
\]

and we call the operators

\[
\mathcal{L}_j = - \sum_{1 \leq i \leq n_j} (W^i_j)^2
\]

the partial sub-Laplacian on \( \mathbf{w}_j \), respectively.

Then \( \mathcal{L} \) maps \( \mathcal{A} \cap C^\infty(G) \), and the spectral properties of each function \( \varphi^G_{\alpha_j, \lambda_j} \) imply

\[
\hat{\mathcal{L}}_j \hat{f}(\alpha, \lambda) = (2\alpha_j + n_j)|\lambda_j|\hat{f}(\alpha, \lambda), \tag{4.11a}
\]

and

\[
\hat{\mathcal{L}} \hat{f}(\alpha, \lambda) = \left( \sum_{j=1}^{N} (2\alpha_j + n_j)|\lambda_j| \right)\hat{f}(\alpha, \lambda). \tag{4.11b}
\]

For \( Z \in \mathfrak{z} \),

\[
\hat{Z} \hat{f}(\alpha, \lambda) = i\lambda(Z)\hat{f}(\alpha, \lambda) = i \sum_{j=1}^{N} \lambda_j(Z_j)\hat{f}(\alpha, \lambda), \tag{4.11c}
\]

where each \( Z_j \in \mathfrak{z}_j \) and \( Z = (Z_1, \ldots, Z_N) \).

We then fix an orthonormal basis \( Z^1_j, \ldots, Z^{d_j}_j \) for each \( \mathfrak{z}_j \). We then write each \( \lambda \in \mathfrak{z}^* \) as

\[
\lambda = (\lambda^1, \ldots, \lambda^N),
\]

and each \( \lambda^i \) is written in the dual basis of \( Z^1_j, \ldots, Z^{d_j}_j \); i.e.

\[
\lambda^i = (\lambda^i_1, \ldots, \lambda^i_{d_j}),
\]

with \( \lambda^i_j = \lambda^i(Z^i_j) \in \mathbb{R} \). We then define formally the operators \( T_j \) on functions on \( \exp \mathfrak{z}_j \) as

\[
T_j = (Z^1_j, \ldots, Z^{d_j}_j),
\]
and we will write $-iT_j$ for $(-iZ_j^1, \ldots, -iZ_j^d)$, for simplicity of notation.

The operators $L_j$ and $-iT_j$ are formally self-adjoint in $\mathcal{S}(G)$ and commute with each other. Hence, via the spectral theorem, we define the operators

$$m(\mathcal{L}_1, \ldots, \mathcal{L}_N, -iT_1, \ldots, -iT_N)$$

for any function $m \in L^\infty(\mathbb{R}_+^N, \mathbb{R}^d)$, and the equations (4.11) imply that

$$(m(\mathcal{L}_1, \ldots, \mathcal{L}_N, -iT_1, \ldots, -iT_N)f) \hat{\cdot}(\alpha, \lambda) = m((2\alpha_1 + n_1)|\lambda_1|, \ldots, (2\alpha_N + n_N)|\lambda_N|, \lambda) \hat{f}(\alpha, \lambda) \quad (4.12)$$

for any polyradial function $f$.

We also define, by means of the spectral theorem, the operators

$$m(|T_1|^{-1}\mathcal{L}_1, \ldots, |T_N|^{-1}\mathcal{L}_N, -iT_1, \ldots, -iT_N),$$

which then satisfy

$$(m(|T_1|^{-1}\mathcal{L}_1, \ldots, |T_N|^{-1}\mathcal{L}_N, -iT_1, \ldots, -iT_N)f) \hat{\cdot}(\alpha, \lambda) = m(2\alpha_1 + n_1, \ldots, 2\alpha_N + n_N, \lambda) \hat{f}(\alpha, \lambda). \quad (4.13)$$

The purpose of the remaining chapters of this thesis is the study of the boundedness of the operators (4.12) and (4.13), given regularity conditions on the function $m$.

### 4.3 Littlewood-Paley theory

In this section we develop a Littlewood-Paley theory for the group $G = G_1 \times \cdots \times G_N$. As in Chapter 2 of this thesis, we follow [MRS96], and start by fixing a smooth cutoff
function $\phi \in C^\infty(\mathbb{R})$, $\phi \geq 0$, supported in $(1/2, 2)$, and such that

$$\sum_{j \in \mathbb{Z}} \phi(2^j|x|)^2 = 1, \quad x \neq 0.$$  

We then set, for $j \in \mathbb{Z}, x \in \mathbb{R}^N$,

$$\phi_j(x) = \phi(2^{-j}(x_1 + \ldots + x_N)),$$

and, for

$$l = (l_1, \ldots, l_N) \in \mathbb{Z}^{d_1} \times \cdots \times \mathbb{Z}^{d_N} = \mathbb{Z}^d, \quad y = (y_1, \ldots, y_N) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N} = \mathbb{R}^d,$$

we define

$$\tilde{\phi}_l(y) = \phi(2^{-l_1^1} |y_1^1|) \phi(2^{-l_2^2} |y_2^2|) \cdots \phi(2^{-l_N^{d_N}} |y_N^{d_N}|).$$

Note that, since the support of the function is away from 0, the functions $\phi_j$ and $\tilde{\phi}_l$ are smooth and

$$\sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^d} \phi_j(x)^2 \tilde{\phi}_l(y)^2 = 1, \quad (4.14)$$

for $(x, y) \in \mathbb{R}_+ \times \mathbb{R}^d, y_i \neq 0$.

For each $j \in \mathbb{Z}$, let $\varphi_j$ be the kernel of the operator $\phi_j(\mathcal{L})$ on $G$,

$$\varphi_j = \phi_j(\mathcal{L}) \delta_{(0,0)},$$

where $\delta_{(0,0)}$ is the Dirac-delta distribution centered at the identity $(0,0)$ of the group $G$. We also write $\psi_l$, for $l \in \mathbb{Z}^d$, the kernel of the operator

$$\tilde{\phi}_l(-iT_1, \ldots, -iT_N).$$
Note that
\[ \varphi_j \ast \psi_l = (\phi_j(L) \otimes \tilde{\phi}_l(-iT_1, \ldots, -iT_N)) \delta_{(0,0)}. \]

We then define the \( g \)-function
\[ g_1(f)(w, z) = \left( \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |f \ast (\varphi_j \ast \psi_l)(w, z)|^2 \right)^{1/2}, \tag{4.15} \]
for \( f \in \mathcal{S}(G) \). The Plancherel formula, together with (4.14), implies that \( g_1 \) extends to an isometry on \( L^2(G) \). We now have the following proposition.

**Proposition 4.1.** For \( 1 < p < \infty \), there exists \( c_p > 0 \) such that
\[ c_p^{-1} ||f||_{L^p} \leq ||g_1(f)||_{L^p} \leq c_p ||f||_{L^p}. \]

**Proof.** The proof of this proposition also follows as the one for Proposition 4.1 in [MRS96]. By standard duality and randomization arguments it is sufficient to prove that the functions
\[ m_\varepsilon(x, y) = \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^d} \varepsilon_j \varepsilon_l^{1,1} \cdots \varepsilon_l^{1,d_1} \cdots \varepsilon_l^{N,d_N} \phi_j(x) \tilde{\phi}_l(y), \tag{4.16} \]
\( m_\varepsilon \) satisfy Hörmander-Mihlin conditions of any degree, uniformly in the \((d+1)\)-multi sequences \( \varepsilon = (\varepsilon_j, \varepsilon_l^{1,1}, \ldots, \varepsilon_l^{1,d_1}, \ldots, \varepsilon_l^{N,d_N}) \) with \( \varepsilon^*_\varepsilon = \pm 1 \).

As the multiple sum (4.16) is separable, we can simply verify such conditions on the functions
\[ \sum_{j \in \mathbb{Z}} \varepsilon_j \phi_j(x) \quad \text{and} \quad \sum_{l \in \mathbb{Z}} \delta_l \phi(2^{-l} |y_k^p|), \]
for \( k = 1, \ldots, N, p = 1, \ldots, d_k \) and sequences \( (\varepsilon_j) \) and \( (\delta_l) \) of \( \pm 1 \). For the former, for
any $N$-multiindex $\alpha$,

$$\left| x^{\alpha} \partial^{\alpha}_{x} \left( \sum_{j \in \mathbb{Z}} \varepsilon_{j} \phi_{j}(x) \right) \right| = \left| \sum_{j \in \mathbb{Z}} \varepsilon_{j} x^{\alpha} \partial^{\alpha}_{x} \left( \phi_{j}(x) \right) \right| = \left| \sum_{j \in \mathbb{Z}} \varepsilon_{j} x^{\alpha} 2^{-j|\alpha|} \phi_{j}^{(|\alpha|)}(x) \right|$$

$$\leq \sum_{j \in \mathbb{Z}} |x|^{\alpha} 2^{-j|\alpha|} |\phi_{j}^{(|\alpha|)}(x)| \leq C_{\alpha},$$

for $C_{\alpha}$ independent of the sequence $(\varepsilon_{j})$, since $|x| \sim 2^{j}$ in the support of $\phi_{j}(x)$. We have written $\phi_{j}^{(n)}(x)$ for $\phi^{(n)}(2^{-j}(x_{1} + \ldots + x_{N}))$.

Note the we can interchange summation with derivatives, as all of them converge uniformly for every $\alpha$.

Now, similarly, for every $d$-multiindex $\alpha = (\alpha_{1}^{d}, \ldots, \alpha_{N}^{d})$,

$$y^{\alpha} \partial^{\alpha}_{y} \left( \sum_{l \in \mathbb{Z}} \delta_{j} \phi(2^{-l}|y_{k}|) \right) = \sum_{l \in \mathbb{Z}} \delta_{j} y^{\alpha} \partial^{\alpha}_{y} \left( \phi(2^{-l}|y_{k}|) \right)$$

$$= \sum_{l \in \mathbb{Z}} \delta_{j} \cdot |y_{k}|^{\alpha_{k}} \cdot 2^{-l|\alpha_{k}|} \cdot \phi^{(\alpha_{k})}(2^{-l}|y_{k}|),$$

provided every $\alpha_{q} = 0$ if $q \neq k$ and $r \neq p$, and equal to 0 otherwise. Therefore

$$\left| y^{\alpha} \partial^{\alpha}_{y} \left( \sum_{l \in \mathbb{Z}} \delta_{j} \phi(2^{-l}|y_{k}|) \right) \right| \leq C_{\alpha},$$

independently of the sequence $(\delta_{l})$. \(\square\)

We now consider the operator $\phi_{j} \left( \sum_{k=1}^{N} |T_{k}|^{-1} L_{k} \right)$, defined via the spectral theorem, and let $\Phi_{j}$ be its kernel. We then define the second $g$-function

$$g_{2}(f)(w, z) = \left( \sum_{j \in \mathbb{Z}, l \in \mathbb{Z}^{d}} \left| f * (\Phi_{j} * \psi_{l})(w, z) \right|^{2} \right)^{1/2}, \quad (4.17)$$

defined, say, for $f \in \mathcal{S}(G)$. As above, $g_{2}$ is an isometry on $L^{2}(G)$. We have the
following proposition.

**Proposition 4.2.** For $1 < p < \infty$, there exists $c'_p > 0$ such that

$$c_p^{-1} \|f\|_{L^p} \leq \|g_2(f)\|_{L^p} \leq c'_p \|f\|_{L^p}.$$  

**Proof.** As in the proof of Proposition 4.1, it is sufficient to prove that the functions

$$m_{\varepsilon}(x, y) = \sum_{j \in \mathbb{Z}} \varepsilon_{j}^{0,1} \cdots \varepsilon_{l_1}^{1,d_1} \cdots \varepsilon_{l_N}^{N,d_N} \phi_j \left( \frac{x_1}{|y_1|}, \ldots, \frac{x_N}{|y_N|} \right) \tilde{\phi}_l(y)$$

satisfy Hörmander-Mihlin conditions of any degree, uniformly in the $(d + 1)$-multi sequences $\varepsilon = (\varepsilon_{j}^{0,1}, \ldots, \varepsilon_{l_1}^{1,d_1}, \ldots, \varepsilon_{l_N}^{N,d_N})$ with $\varepsilon^* = \pm 1$.

However, this time the sum defining $m_{\varepsilon}$ is not separable, but it still converges uniformly away from zero, together with the sums of the derivatives.

We observe that

$$\frac{\partial}{\partial x_k} \left( \phi_j \left( \frac{x_1}{|y_1|}, \ldots, \frac{x_N}{|y_N|} \right) \tilde{\phi}_l(y) \right) = 2^{-j} \frac{x_k}{|y_k|} \phi'_j \left( \frac{x_1}{|y_1|}, \ldots, \frac{x_N}{|y_N|} \right) \tilde{\phi}_l(y)$$

and

$$\frac{\partial}{\partial y_k} \left( \phi_j \left( \frac{x_1}{|y_1|}, \ldots, \frac{x_N}{|y_N|} \right) \tilde{\phi}_l(y) \right) = 2^{-j} \frac{x_k}{|y_k|} \left( \frac{y_k}{|y_k|^2} \right) \phi'_j \left( \frac{x_1}{|y_1|}, \ldots, \frac{x_N}{|y_N|} \right) \tilde{\phi}_l(y)$$

$$+ \phi_j \left( \frac{x_1}{|y_1|}, \ldots, \frac{x_N}{|y_N|} \right) \phi(2^{-l_1} y_1^k) \cdots 2^{-l_i} \phi'(2^{-l_k} y_k^i) \cdots \phi(2^{-l_N} y_N^d),$$

hence

$$f(x, y) = x_k \frac{\partial}{\partial x_k} \left( \phi_j \left( \frac{x_1}{|y_1|}, \ldots, \frac{x_N}{|y_N|} \right) \tilde{\phi}_l(y) \right) = 2^{-j} \frac{x_k}{|y_k|} \phi'_j \left( \frac{x_1}{|y_1|}, \ldots, \frac{x_N}{|y_N|} \right) \tilde{\phi}_l(y).$$
\[ g(x, y) = y^i_k \frac{\partial}{\partial y^i_k} \left( \phi_j \left( \frac{x_1}{y_1}, \ldots, \frac{x_N}{y_N} \right) \tilde{\phi}_l(y) \right) \\
= 2^{-j} \frac{x_k}{|y_k|} \left( - \frac{(y^i_k)^2}{|y_k|^2} \right) \phi_j' \left( \frac{x_1}{y_1}, \ldots, \frac{x_N}{y_N} \right) \tilde{\phi}_l(y) \\
+ \phi_j \left( \frac{x_1}{y_1}, \ldots, \frac{x_N}{y_N} \right) \phi(2^{-l_1} y^1_1) \cdots (2^{-l_k} y^i_k) \phi' \left( 2^{-l_k} y^i_k \right) \cdots \phi(2^{-l_N} y^d_N). \]

In the support of \( \phi_j \left( \frac{x_1}{y_1}, \ldots, \frac{x_N}{y_N} \right) \tilde{\phi}_l(y) \),

\[ 2^{j-1} \leq \frac{x_1}{|y_1|} + \ldots + \frac{x_N}{|y_N|} \leq 2^{j+1}, \quad 2^{l_k-1} \leq |y^i_k| \leq 2^{l_k+1}, \quad (4.18) \]

so \( 0 \leq 2^{-j} \frac{x_k}{|y_k|} \leq 2 \) and \( \frac{1}{2} \leq |2^{-l_k} y^i_k| \leq 2 \), together with \( \frac{|y^i_k|^2}{|y_k|^2} \leq 1 \), and hence both \( f(x, y) \) and \( g(x, y) \) are linear combinations of functions of the form

\[ \theta_j^{(1)} \left( \frac{x_1}{y_1}, \ldots, \frac{x_N}{y_N} \right) \eta_l^{(1)}(y), \]

where \( \theta_j^{(1)} \) and \( \eta_l^{(1)} \) are smooth functions supported in \( (4.18) \), and uniformly bounded by a constant multiple of

\[ ||\phi||_{C^1} = \sup_{t \in \mathbb{R}} \sup_{0 \leq j \leq 1} |\phi^{(j)}(t)|. \]

We therefore can then prove inductively that

\[ (x \partial_x)^\alpha (y \partial_y)^\beta \left( \phi_j \left( \frac{x_1}{y_1}, \ldots, \frac{x_N}{y_N} \right) \tilde{\phi}_l(y) \right), \]

for an \( N \)-multiindex \( \alpha = (\alpha_1, \ldots, \alpha_N) \) and a \( d \)-multiindex \( \beta = (\beta_1 \ldots \beta^d_N) \), is a linear combination of terms of the form

\[ \theta_j^{(\alpha + |\beta|)} \left( \frac{x_1}{y_1}, \ldots, \frac{x_N}{y_N} \right) \eta_l^{(|\beta|)}(y), \quad (4.19) \]
with coefficients depending only on $\alpha$ and $\beta$, where $\theta^{(|\alpha|+|\beta|)}_j$ and $\eta^{(\beta)}_l$ are smooth functions supported in (4.18), and uniformly bounded by

$$||\phi||_{C^{(|\alpha|+|\beta|)}} \quad \text{and} \quad ||\phi||_{C^{(\beta)}},$$

respectively.

As only a fixed finite number of terms, for each $(x,y)$, are nonzero in the sum defining $m_\varepsilon$, we therefore have

$$\sup_{x \in \mathbb{R}^N, y \in \mathbb{R}^d} \left| (x \partial_x)^\alpha (y \partial_y)^\beta m_\varepsilon (x,y) \right| \leq C_{\alpha,\beta}$$

for every multiindices $\alpha, \beta$. \qed
Chapter 5

Multiplier operators on products of $H$-type groups

5.1 Multiplier theorem

In this chapter we assume that $G = G_1 \times \cdots \times G_N$ is the product of $N$ $H$-type groups $G_j$, each of dimension $2n_j + d_j$, $n = n_1 + \ldots + n_N$ and $d = d_1 + \ldots + d_N$, so that $G$ has dimension $2n + d$, and we take the same notation as in the previous chapter.

Let $m$ be a bounded continuous function defined on the set

$$(2N + n_1) \times (2N + n_2) \times \cdots \times (2N + n_N) \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_N}$$

$$= \{(2\alpha_1 + n_1, \ldots, 2\alpha_N + n_N, \lambda_1, \ldots, \lambda_N) : \alpha_j \in \mathbb{N}, \lambda_j \in \mathbb{R}^{d_j}\}.$$

Then, by the spectral theorem and the Fourier theory on $G$ we can define the operator

$$\mathcal{T} = m(|T_1|^{-1} \mathcal{L}_1, \ldots, |T_N|^{-1} \mathcal{L}_N, -iT_1, \ldots, -iT_N). \quad (5.1)$$

Recall the have fixed orthonormal bases $Z_j^{1}, \ldots, Z_j^{d_j}$ for each $j$, and that each $T_j$
is given by
\[ T_j = (Z^1_j, \ldots, Z^d_j). \]

Define \( \tilde{m} \) on \( \mathbb{Z}^N \times \mathbb{R}^d \) by
\[
\tilde{m}(\alpha, \lambda) = \begin{cases} 
  m(2\alpha_1 + n_1, \ldots, 2\alpha_N + n_N, \lambda_1, \ldots, \lambda_N) & \alpha_k \geq 0, \lambda_k \in \mathbb{R}^{d_k}, 1 \leq k \leq N \\
  0 & \text{otherwise.}
\end{cases}
\]

We have the following theorem.

**Theorem 5.1.** Suppose that, for some \( s_k > n_k, s^j_k > 1/2, k = 1, \ldots, N, j = 1, \ldots, d_k \), the function \( \tilde{m} \) is in \( \mathcal{L}_{s,s',\text{loc}}(\mathbb{Z}^N \times \mathbb{R}^d) \), where \( s = (s_1, \ldots, s_N) \) and \( s' = (s^1_1, s^2_1, \ldots, s^d_1, \ldots, s^d_N) \). Then, the operator (5.1) is bounded in \( L^p(G) \), with norm
\[
||T||_{L^p \rightarrow L^p} \lesssim ||\tilde{m}||_{\mathcal{L}_{s,s',\text{loc}}}. \tag{5.2}
\]

The constant in (5.2) depends only on \( G, s, s' \), and \( p \).

The multi-parameter scale invariant localized Sobolev space \( \mathcal{L}_{\alpha,\beta,\text{loc}}(\mathbb{Z}^N \times \mathbb{R}^d) \) on \( \mathbb{Z}^N \times \mathbb{R}^d \) is defined analogously as in (2.4) on \( \mathbb{Z}^n \times \mathbb{R}^n \): Fix \( \eta_0 \in C_0^\infty \) of compact support in \( \mathbb{R}_+ \), with \( \eta_0 \equiv 1 \) on \( (1/2, 2) \), and define \( \eta \) on \( \mathbb{R}^N \times \mathbb{R}^d \) by \( \eta(x,y) = \eta_1(x)\eta_2(y) \), where
\[
\eta_1(x) = \begin{cases} 
  \eta_0(x_1 + \ldots + x_N) & x_1, \ldots, x_N > 0 \\
  0 & \text{otherwise},
\end{cases}
\] 
\[ \eta_2(y) = \eta_0(|y_1^1|) \cdots \eta_0(|y^d_N|). \tag{5.3} \]

Given a \( (d+1) \)-tuple \( r = (r_0, r^1_1, \ldots, r^d_1, \ldots, r^d_N) \) of positive numbers, we define the functions
\[
\eta^r(x,y) = \eta(r_0 x, r^1_1 y_1, \ldots, r^d_N y^d_N) \tag{5.4}
\]
\[ \eta_r(x,y) = \eta(x/r_0, y_1/r^1_1, \ldots, y^d_N/r^d_N). \]
on $\mathbb{R}^N \times \mathbb{R}^d$. We then define, for an $N$-tuple $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $d$-tuple $\beta = (\beta_1^1, \ldots, \beta_N^d)$ of positive numbers, the space $L_{\alpha, \beta, \text{loc}}(\mathbb{Z}^N \times \mathbb{R}^d)$ as the set of functions $f$ on $\mathbb{Z}^N \times \mathbb{R}^d$ such that

$$
||f||_{L_{\alpha, \beta, \text{loc}}} = \sup_{r \in \mathbb{R}_{d+1}^+} (r_0^N r_1^1 \cdots r_1^n)^{1/2} \left( \prod_{k=1}^N (1 + r_0^k |\Delta_k|) \right)^{\alpha_k} \prod_{j=1}^d (1 + r_0^j |\Delta_j| + |r_0^j \partial_{(k,j)}|)^{\beta_j^k} \eta_r \right)_{l^2(L^2)},
$$

(5.5)

where, as before, $|| \cdot ||_{l^2(L^2)}$ denotes the norm of the Hilbert space $l^2(\mathbb{Z}^N, L^2(\mathbb{R}^d))$ of functions on $\mathbb{Z}^N \times \mathbb{R}^d$, and $\Delta_k$ and $\partial_{(k,j)}$ denote the partial difference and differential operators, on the $k$th discrete and the $(k,j)$th continuous variables, respectively.

The spaces $L_{\alpha, \beta, \text{loc}}(\mathbb{Z}^N \times \mathbb{R}^d)$ share all the properties with the respective spaces on $\mathbb{Z}^n \times \mathbb{R}^n$ discussed in Section 2.1. We will make use of this properties, as well as the analogous versions of Propositions 2.2 and 2.3, without stating them explicitly. The proofs of the following corollaries also follow as the proofs of Corollaries 2.5 and 2.6.

Given an $N$-tuple $s = (s_1, \ldots, s_N)$ and a $d$-tuple $s' = (s_1^1, s_1^2, \ldots, s_1^d, s_2^1, \ldots, s_N^d)$ of positive numbers, we define the space $L_{s, s', \text{loc}}(\mathbb{R}_+^N \times \mathbb{R}^d)$ as the set of functions $f$ on $\mathbb{R}_+^N \times \mathbb{R}^d$ such that

$$
||f||_{L_{s, s', \text{loc}}} = \sup_{r \in \mathbb{R}_{d+1}^+} ||f^r \eta||_{L^2(\mathbb{L}_2)},
$$

where, as before,

$$f^r(x, y) = f(r_0 x, r_0^1 y_1^1, \ldots, r_N^d y_N^d)$$

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and

$$\|f\|_{L^2(L^2_x)}^2 = \int_{R^N \times R^d} \prod_{k=1}^N (1 + |2\pi \xi_k|)^{2s_k} \prod_{j=1}^{d_k} (1 + |2\pi \zeta_k^j|)^{2s_k^j} |\hat{f}(\xi, \zeta)|^2 d\xi d\zeta.$$ 

We observe that, if all $s_k, s_k^j$ are nonnegative integers, then

$$\|f\|_{L^2(L^2_x)}^2 \approx \sum_{\alpha, \beta} \|\partial_\alpha^\beta \partial_y^\beta f\|_{L^2(R^N \times R^d)}^2,$$

where, for the $d$-multiindex $\beta = (\beta_1^1, \ldots, \beta_N^d)$, $\bar{\beta}$ is the $N$-multiindex

$$\bar{\beta} = (|\beta_1|, \ldots, |\beta_N|),$$

with $|\beta_k| = \beta_1^1 + \ldots + \beta_d^k$.

**Corollary 5.2.** Let $m$ be a bounded continuous function on $R^N_+ \times R^d$ and suppose that, for some $s_k > n_k, s_k^j > 1/2$, $m \in L_{s,s',\text{loc}}(R^N_+ \times R^d)$. Then the operators $T$ and $M$, given by (5.1) and

$$M = m(L_1, \ldots, L_N, -iT_1, \ldots, -iT_N),$$

are bounded in $L^p(G), 1 < p < \infty$, with norms

$$\|T\|_{L^p \to L^p} \lesssim \|m\|_{L_{s,s',\text{loc}}},$$

$$\|M\|_{L^p \to L^p} \lesssim \|m\|_{L_{s,s',\text{loc}}}.$$ 

**Proof.** The boundedness of the operator $T$ follows from the fact that if $m$ is continuous
and \( m \in \mathcal{L}_{s,s',\text{loc}}(\mathbb{R}_+^N \times \mathbb{R}^d) \), then function \( \tilde{m} \) defined as above on \( \mathbb{Z}^N \times \mathbb{R}^d \) by

\[
\tilde{m}(\alpha, \lambda) = \begin{cases} 
  m(2\alpha_1 + n_1, \ldots, 2\alpha_N + n_N, \lambda_1, \ldots, \lambda_N) & \alpha_k \geq 0, \lambda_k \in \mathbb{R}^d, 1 \leq k \leq N \\
  0 & \text{otherwise},
\end{cases}
\]

belongs to the space \( \mathcal{L}_{s,s',\text{loc}}(\mathbb{Z}^N \times \mathbb{R}^d) \). But this follows directly from Proposition 2.1.

For the boundedness of \( \mathcal{M} \), we need to prove that the function, defined for \( (x, y) \in \mathbb{R}^N \times \mathbb{R}^d \) by

\[
\tilde{m}(x, y) = m(x_1|y_1|, \ldots, x_N|y_N|, y_1^1, \ldots, y_N^1),
\]

is in \( \mathcal{L}_{s,s',\text{loc}}(\mathbb{R}_+^N \times \mathbb{R}^d) \) if \( m \in \mathcal{L}_{s,s',\text{loc}}(\mathbb{R}_+^N \times \mathbb{R}^d) \).

We consider the case where the \( s_k \) and \( s_k^j \) are integers. Note that

\[
\frac{\partial}{\partial x_k}(\tilde{m}^r \eta)(x, y) = \frac{\partial}{\partial x_k}(m(r_0 x_1|y_1|, \ldots, r_0 x_N|y_N|, r_1^1 y_1^1, \ldots, r_N^1 y_N^1)\eta(x, y))
\]

\[
= r_0 |y_k| (\partial_{x_k} m)(r_0 x_1|y_1|, \ldots, r_0 x_N|y_N|, r_1^1 y_1^1, \ldots, r_N^1 y_N^1)\eta(x, y)
\]

\[
+ m(r_0 x_1|y_1|, \ldots, r_0 x_N|y_N|, r_1^1 y_1^1, \ldots, r_N^1 y_N^1)\partial_{x_k} \eta(x, y).
\]

Thus, one can prove by induction that, for an \( N \)-multiindex \( \alpha = (\alpha_1, \ldots, \alpha_N) \),

\[
\frac{\partial^\alpha}{\partial x^\alpha}(\tilde{m}^r \eta)(x, y) = \sum_{0 \leq \gamma_k \leq \alpha_k} C_{\gamma} r_0^{|\gamma|} |y|^\gamma \cdot (\partial_x^\gamma m)(r_0 x_1|y_1|, \ldots, r_0 x_N|y_N|, r_1^1 y_1^1, \ldots, r_N^1 y_N^1) \cdot \partial_x^{\alpha-\gamma} \eta(x, y),
\]

where we have written \( |y|^\gamma \) for \( |y_1|^\gamma_1 \cdots |y_N|^\gamma_N \). Similarly, we observe that

\[
\frac{\partial}{\partial y_k}(\tilde{m}^r \eta)(x, y) = r_0 x_k \frac{y_k^j}{|y_k|} (\partial_{y_k} m)(r_0 x_1|y_1|, \ldots, r_0 x_N|y_N|, r_1^1 y_1^1, \ldots, r_N^1 y_N^1)\eta(x, y)
\]

\[
+ r_k^j (\partial_{y_k} m)(r_0 x_1|y_1|, \ldots, r_0 x_N|y_N|, r_1^1 y_1^1, \ldots, r_N^1 y_N^1)\eta(x, y)
\]

\[
+ m(r_0 x_1|y_1|, \ldots, r_0 x_N|y_N|, r_1^1 y_1^1, \ldots, r_N^1 y_N^1)(\partial_{y_k} \eta)(x, y),
\]

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and thus we have that $\partial_\alpha^+ \partial_\beta^\alpha (\tilde{m}^\tau \eta)(x, y)$, for an $N$-multiindex $\alpha$ and a $d$-multiindex $\beta$, is a linear combination of terms of the form

$$P_{\gamma, \zeta, \sigma, \rho} \cdot r_0^{\gamma^+} \cdot (\partial_x^\gamma \partial_y^\alpha \tilde{m})(r_0 x_1, \ldots, r_0 x_N, |y_N|, r_1^1 y_1, \ldots, r_N^N y_N) \cdot \partial_x^\sigma \partial_y^\theta \eta(x, y),$$

(5.6)

where $\gamma, \zeta, \rho$ are $N$-multiindexes and $\sigma, \theta$ $d$-multiindexes such that

$$\gamma_k + \rho_k \leq \alpha_k, \quad \sigma_k^j + \theta_k^j \leq \beta_k^j, \quad \text{and} \quad \zeta_k + |\sigma_k| \leq |\beta_k|,$$

and the $P_{\gamma, \zeta, \sigma, \rho}$ are polynomials in the $x_k, y_k, |y_k|$ and $|y_k|^{-1}$. Now, in the support of $\eta$,

$$0 < x_k \leq 2, \quad \text{and} \quad \frac{1}{2} \leq |y_k| \leq 2,$$

so the $P_{\gamma, \zeta, \sigma, \rho}$ are uniformly bounded and the $L^2$ norms of the functions (5.6) are estimated by

$$\int |r_0^{\gamma^+} \cdot (\partial_x^\gamma \partial_y^\alpha \tilde{m})(r_0 x_1, \ldots, r_0 x_N, |y_N|, r_1^1 y_1, \ldots, r_N^N y_N) \cdot \partial_x^\sigma \partial_y^\theta \eta(x, y)|^2 \, dx \, dy \lesssim \int |\partial_x^\gamma \partial_y^\alpha \tilde{m}(x, y) \cdot \partial_x^\sigma \partial_y^\theta \tilde{\eta}(x, y)|^2 \, dx \, dy \lesssim \int |\partial_x^{\gamma^+} \partial_y^\alpha \tilde{m}(x, y) \cdot \partial_x^\sigma \partial_y^\theta \tilde{\eta}(x, y)|^2 \, dx \, dy,$$

where $\tilde{\eta}$ is a $C^\infty$ function of the form $\tilde{\eta}_0(x_1 + \ldots + x_N) \eta_0(y_1^1 \cdots y_N^d N)$, with $\tilde{\eta}_0$ supported away from zero (say, on $(1/4, 4)$), since

$$\frac{x_1}{|y_1|} + \ldots + \frac{x_N}{|y_N|} \sim x_1 + \ldots + x_N \quad \text{in the support of } \eta.$$
such that $0 \leq \alpha_k \leq s_k$ and $0 \leq \beta_{k}^{j} \leq s_{j}^{k}$, we obtain

$$||\hat{m}||_{L_{s,s',\text{loc}}^{s}((\mathbb{R}^{N} \times \mathbb{R}^{d}))} \lesssim ||m||_{L_{s,s',\text{loc}}^{s}((\mathbb{R}^{N} \times \mathbb{R}^{d}))}$$

for integer $s_k, s_j^k$. For general $s_k, s_j^k$ we proceed by interpolation. \qed

**Corollary 5.3.** Let $m$ be a bounded continuous function on $\mathbb{R}$ such that, for some $s_0 > (2n + d)/2$, $m \in L_{s_0,\text{loc}}^{2}(\mathbb{R})$. Then $m$ is a spectral multiplier, i.e. the operator $m(\mathcal{L})$ is bounded in $L^{p}(G)$.

**Proof.** As in the proofs of Propositions 2.2 and 2.3, we can prove that the function

$$m_0(x,y) = \begin{cases} 
    m(x_1 + \ldots + x_N) & x_k \in \mathbb{R}^+, y_k \in \mathbb{R}^{d_k}, 1 \leq k \leq N \\
    0 & \text{otherwise},
\end{cases}$$

is in some $L_{s,s',\text{loc}}^{s}((\mathbb{R}^{N} \times \mathbb{R}^{d}))$, with $s_k > n_k$, $s_j^k > 1/2$ and $|s| + |s'| = s_0$, since $s_0 > (2n + d)/2$. Therefore, by Corollary 5.2, the operator

$$m(\mathcal{L}) = m_0(\mathcal{L} _1, \ldots, \mathcal{L} _N, -iT_1, \ldots, -iT_N)$$

is bounded in $L^{p}(G)$. \qed

### 5.2 Proof of Theorem 5.1

As in the case of Theorem 2.4, the proof of Theorem 5.1 follows by the following lemma.

Define, for $j \in \mathbb{Z}$, $l \in \mathbb{Z}^{d}$,

$$m_{j,l}(x,y) = m(x,y)\eta_{r(j,l)}(x,y),$$
for \( x \in \mathbb{R}_+^N, y \in \mathbb{R}^d \), where \( \eta_r \) is defined as in (5.4) and \( r(j,l) \) is the \((d + 1)\)-tuple \((2^j, 2^{l_1}, \ldots, 2^{l_d})\). Set \( M_{j,l} \) to be the kernel of the operator

\[
m_{j,l}(|T_1|^{-1}L_1, \ldots, |T_N|^{-1}L_N, -iT_1, \ldots, -iT_N).
\]

**Lemma 5.4.** Under the hypotheses of Theorem 5.1, for \( j \in \mathbb{Z}, l \in \mathbb{Z}^d \), we have the estimate

\[
\int_G \left| \prod_{k=1}^N (1 + 2^j |L_k|)^{2n_k} \prod_{i=1}^{d_k} (1 + 2^l_i |L_k|) M_{j,l}(w,z) \right|^2 \, dw \, dz \lesssim 2^{L_j n - jN} \times \\
\sum_{\alpha \in \mathbb{Z}^N} \int_J \left| \prod_{k=1}^N (1 + 2^j |\Delta_k|)^{2n_k} \prod_{i=1}^{d_k} (1 + 2^l_i |\Delta_k| + 2^l_i |\partial_{k,i}|) \tilde{m}_{j,l}(\alpha, \lambda) \right|^2 \, d\lambda,
\]

where \( L_k = \max_{1 \leq i \leq d_k} l_k^i \), \( L = n_1 L_1 + \ldots + n_N L_N \) and the constant of inequality (5.7) is independent of \( m, j, \) and \( l \).

The function \( \tilde{m}_{j,l} \) on \( \mathbb{Z}^N \times \mathbb{R}^d \) is defined as \( \tilde{m} \) above:

\[
\tilde{m}_{j,l}(\alpha, \lambda) = \begin{cases} 
m_{j,l}(2\alpha_1 + n_1, \ldots, 2\alpha_N + n_N, \lambda) & \text{if } \alpha_1, \ldots, \alpha_N \geq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Note that, as each \( 2\alpha_k + n_k \) is a positive integer, \( \tilde{m}_{j,l}(\alpha, \lambda) = 0 \) unless \( j \geq 0 \).

Assuming Lemma 5.4, we proceed with the proof of Theorem 5.1. First, the Plancherel formula (4.9) implies that

\[
\int_G |M_{j,l}(w,z)|^2 \, dw \, dz = C \sum_{\alpha \in \mathbb{Z}^N} \int_J |\tilde{m}_{j,l}(\alpha, \lambda)|^2 |\lambda_1|^{n_1} \cdots |\lambda_N|^{n_N} \, d\lambda.
\]

Since, in the support of \( m_{j,l} \), \( j \geq 0 \), \( |\lambda_k| \sim 2^{l_k} \) and

\[
|\lambda_k| \sim \max_{1 \leq i \leq d_k} |\lambda_k^i| \sim 2^{L_k},
\]

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so we have

\[
\int_G |M_{j,l}(w,z)|^2 \, dw \, dz \sim 2^L \sum_{\alpha \in \mathbb{N}^N} \int \left| \tilde{m}_{j,l}(\alpha, \lambda) \right|^2 \, d\lambda
\]

\[
\leq 2^{L+jn-jN} \sum_{\alpha \in \mathbb{N}^N} \int \left| \tilde{m}_{j,l}(\alpha, \lambda) \right|^2 \, d\lambda,
\]

as \( N \geq n \). Interpolating (5.8) with estimate (5.7) of Lemma 5.4, we obtain the estimate

\[
\int_G \left| \prod_{k=1}^N (1 + 2^j + L_k |w_k|^2)^{1/2+\varepsilon} \prod_{i=1}^{d_k} (1 + 2^l_i |z_k^i|^1 + 2^l_i |\partial_{k,i}|) \right|^{1/2+\varepsilon} \left|\tilde{m}_{j,l}(\alpha, \lambda)\right|^2 \, d\lambda \lesssim 2^{L+j(n-N)} \times
\]

\[
\sum_{\alpha \in \mathbb{Z}^N} \int \left| \prod_{k=1}^N (1 + 2^j |\Delta_k|)^{1+2\varepsilon} \prod_{i=1}^{d_k} (1 + 2^j |\Delta_k| + 2^l_i |\partial_{k,i}|) \right|^{1/2+\varepsilon} \left|\tilde{m}_{j,l}(\alpha, \lambda)\right|^2 \, d\lambda.
\]

By (5.5), we have that

\[
\frac{1}{2^{jN}2^{l_1} \cdots 2^{l_N}}
\]

\[
\sum_{\alpha \in \mathbb{Z}^N} \int \left| \prod_{k=1}^N (1 + 2^j |\Delta_k|)^{1+2\varepsilon} \prod_{i=1}^{d_k} (1 + 2^j |\Delta_k| + 2^l_i |\partial_{k,i}|) \right|^{1/2+\varepsilon} \left|\tilde{m}_{j,l}(\alpha, \lambda)\right|^2 \, d\lambda \lesssim ||\tilde{m}||_{\mathcal{L}_{\gamma,\delta,\text{loc}}}^2,
\]

where the \( N \)-tuple \( \gamma \) and the \( d \)-tuple \( \delta \) are given by

\[
\gamma = (n_1(1+2\varepsilon), \ldots, n_N(1+2\varepsilon)) \quad \text{and} \quad \delta = (1/2 + \varepsilon, \ldots, 1/2 + \varepsilon).
\]

Thus we have the estimate

\[
\int_G \left| \prod_{k=1}^N (1 + 2^j + L_k |w_k|^2)^{1/2+\varepsilon} \prod_{i=1}^{d_k} (1 + 2^l_i |z_k^i|^1 + 2^l_i |\partial_{k,i}|) \right|^{1/2+\varepsilon} M_{j,l}(w,z) \, dw \, dz
\]

\[
\lesssim 2^{L+jn} ||\tilde{m}||_{\mathcal{L}_{\gamma,\delta,\text{loc}}}^2,
\]

(5.9)
where $\bar{l} = l_1^1 + \ldots l_N^d$.

By the Littlewood-Paley theory of Section 4.3, for $f \in L^p(G)$,

$$\|Tf\|_{L^p(G)} \lesssim \|g_2(Tf)\|_{L^p(G)} \lesssim \left\| \left( \sum_{(j,l) \in \mathbb{Z}^N \times \mathbb{Z}^d} |f_{j,l} * M_{j,l}|^2 \right)^{1/2} \right\|_{L^p(G)},$$

where $f_{j,l} = f * (\Phi_j * \psi_l)$. Now, by the Cauchy-Schwarz inequality,

$$|f_{j,l} * M_{j,l}|^2 = \left( \int_G f_{j,l}((w,z)(u,v)^{-1}) M_{j,l}(u,v) dudv \right)^2 \leq \int_G \mathcal{W}(u,v)|M_{j,l}(u,v)|^2 dudv \int_G |f_{j,l}((w,z)(u,v)^{-1})|^2 \mathcal{W}(u,v) dudv,$$

where $\mathcal{W}$ is the weight

$$\mathcal{W}(w,z) = 2^{-(L+jn+i)} \prod_{k=1}^N (1 + 2^{j+k-n} |w_k|^2)^{n_k(1+2\varepsilon)} \prod_{i=1}^{d_k} (1 + 2^{j+i} |z_k^i|)^{1+2\varepsilon}.$$

Thus, by (5.9),

$$|f_{j,l} * M_{j,l}|^2 \lesssim \|\tilde{m}\|_{L^{2,\varepsilon}_{\gamma,\delta,\text{loc}}}^2 \int_G \frac{|f_{j,l}((w,z)(u,v)^{-1})|^2}{\mathcal{W}(u,v)} dudv. \quad (5.10)$$

By a standard duality argument, we can assume $p \geq 2$, so there exists $g \in L^{(p/2)'}(G)$ such that $\|g\|_{L^{(p/2)'}(G)} = 1$ and

$$\left\| \left( \sum_{(j,l) \in \mathbb{Z}^N \times \mathbb{Z}^d} |f_{j,l} * M_{j,l}|^2 \right)^{1/2} \right\|_{L^p(G)} \lesssim 2 \int_G \left( \sum_{(j,l) \in \mathbb{Z}^N \times \mathbb{Z}^d} |f_{j,l} * M_{j,l}(w,z)|^2 \right) g(w,z) dwdz$$

$$\lesssim 2 \|\tilde{m}\|_{L^{2,\varepsilon}_{\gamma,\delta,\text{loc}}}^2 \sum_{(j,l) \in \mathbb{Z}^N \times \mathbb{Z}^d} \int_G \int_G \frac{|f_{j,l}((w,z)(u,v)^{-1})|^2}{\mathcal{W}(u,v)} dudv g(w,z) dwdz,$$

where we have used (5.10). By Fubini’s theorem, the integrals on the right are equal.
\[
\int_G \int_G \left| f_{j,l}((w, z)(u, v)^{-1}) \right|^2 \frac{dudv}{W(u, v)} g(w, z) dwdz
\]
\[
= \int_G \int_G \left| f_{j,l}(u, v) \right|^2 \frac{W((u, v)^{-1}(w, z))}{W(u, v)} dwdz dudv
\]
\[
= \int_G \left| f_{j,l}(u, v) \right|^2 \int_G \frac{g(w, z)}{W((u, v)^{-1}(w, z))} dwdz dudv
\]
so we now estimate the integral \( \int_G \frac{g((u, v)(w, z))}{W((w, z))} dwdz \). Define the sets, for \( p_k, q_k^i \in \mathbb{N}, k = 1, \ldots, N, i = 1, \ldots, d_k \),

\[
A_k(p_k) = \begin{cases} 
\{w_k \in \exp w_k : |w_k|^2 \leq 2^{-j-L_k} \} & p_k = 0, \\
\{w_k \in \exp w_k : 2^{-j-L_k+p_k-1} \leq |w_k|^2 \leq 2^{-j-L_k+p_k} \} & p_k \geq 1,
\end{cases}
\]

and

\[
B_k^i(q_k^i) = \begin{cases} 
\{z_k^i \in \exp z_k^i : |z_k^i| \leq 2^{-l_k} \} & q_k^i = 0, \\
\{z_k^i \in \exp z_k^i : 2^{-l_k+q_k^i-1} \leq |z_k^i| \leq 2^{-l_k+q_k^i} \} & q_k^i \geq 1,
\end{cases}
\]

and \( A(p) = A_1(p_1) \times \cdots \times A_N(p_N) \) and \( B(q) = B_1^1(q_1^1) \times \cdots \times B_N^{d_N}(q_N^{d_N}) \), for respective \( N \)-tuples \( p \) and \( d \)-tuples \( q \) of nonnegative integers. Then, for \( (w, z) \in A(p) \times B(q) \), we have

\[
W(w, z) \sim 2^{-(L+n+1)} \prod_{k=1}^N (1 + 2^{p_k})^{n_k(1+2\varepsilon)} \prod_{i=1}^{d_k} (1 + 2^{l_k})^{1+2\varepsilon}
\]
\[
\geq 2^{-(n_L+\ldots+n_N L_N+n_1j+\ldots+n_N j+l_1+\ldots+l_N^{d_N})} \prod_{k=1}^N 2^{p_k n_k(1+2\varepsilon)} \prod_{i=1}^{d_k} 2^{l_k(1+2\varepsilon)}
\]
\[
= \prod_{k=1}^N 2^{n_k(-j-L_k+p_k)} \prod_{i=1}^{d_k} 2^{-l_k+q_k^i} \times \prod_{k=1}^N 2^{2n_k p_k} \prod_{i=1}^{d_k} 2^{2\varepsilon q_k^i}
\]
and thus

\[
\int_G \frac{g((u,v)(w,z))}{W((w,z))} dwdz \leq \sum_{p \in \mathbb{N}^N, q \in \mathbb{N}^d} \frac{1}{\prod_{k=1}^{N} 2^{2^{2n_k} p_k} \prod_{i=1}^{d_k} 2^{2^{2n_k} q_k}} \int_{A(p) \times B(q)} g((u,v)(w,z)) \prod_{k=1}^{N} 2^{2n_k(-j-L_k+p_k)} \prod_{i=1}^{d_k} 2^{-l_i+q_k} dwdz
\]

\[
\leq \sum_{p \in \mathbb{N}^N, q \in \mathbb{N}^d} \frac{1}{\prod_{k=1}^{N} 2^{2^{2n_k} p_k} \prod_{i=1}^{d_k} 2^{2^{2n_k} q_k}} \mathcal{M}g(u,v) = C \mathcal{M}g(u,v),
\]

where \(\mathcal{M}g(u,v)\) is the strong maximal function

\[
\mathcal{M}g(u,v) = \sup_{r \in \mathbb{R}^N, r' \in \mathbb{R}^d} \frac{1}{r_1^{2n_1} \cdots r_N^{2n_N} \cdot r'_1^{d_1} \cdots r'_d^{d_N}} \int \cdots \int |g((u,v)(w,z))| dwdz,
\]

which is bounded in \(L^p(G), 1 < p < \infty, [\text{Chr92}].\) Therefore, putting together all previous estimates and applying Hölder’s inequality,

\[
||Tf||^2_{L^p(G)} \lesssim ||\tilde{m}||^2_{L_{\gamma,d,\text{loc}}} \sum_{(j,l) \in \mathbb{Z}^N \times \mathbb{Z}^d} \int_G |f_{j,l}(u,v)|^2 \int_G \frac{g((u,v)(w,z))}{W((w,z))} dwdz dudv
\]

\[
\lesssim ||\tilde{m}||^2_{L_{\gamma,d,\text{loc}}} \sum_{(j,l) \in \mathbb{Z}^N \times \mathbb{Z}^d} \int_G |f_{j,l}(u,v)|^2 \mathcal{M}g(u,v) dudv
\]

\[
\leq ||\tilde{m}||^2_{L_{\gamma,d,\text{loc}}} \int_G \sum_{(j,l) \in \mathbb{Z}^N \times \mathbb{Z}^d} |f_{j,l}(u,v)|^2 dudv \mathcal{M}g(u,v)
\]

\[
\leq ||\tilde{m}||^2_{L_{\gamma,d,\text{loc}}} ||g_2(f)||_{L^p(G)} ||\mathcal{M}g||_{L^{(p/2)'}(G)} \lesssim ||\tilde{m}||^2_{L_{\gamma,d,\text{loc}}} ||f||^2_{L^p(G)}.
\]

### 5.3 Proof of Lemma 5.4

We now prove Lemma 5.4, whose proof follows similarly as the proof for Lemma 2.7. However, the homogeneity of the operators \(|T_k|^{-1} \mathcal{L}_k\) or \(-iT_k\) cannot be used this time to simplify the proof, as the dimension of the center of each group \(G_k\) is no longer equal to 1.
By the Plancherel formula, we have to estimate

\begin{equation}
C_{g} \int_{\prod_{k=1}^{N} z_k \setminus \{0\}} \sum_{\alpha \in \mathbb{N}^N} \left| \prod_{k=1}^{N} (1 + 2^{j + L_k} \partial_{|w_k|^2})^{n_k} \prod_{\nu=1}^{d_k} (1 + 2^{j \nu} \partial_{iz_k^\nu}) \tilde{m}_{j,\nu}(\alpha, \lambda) \right|^2 D_G(\lambda) d\lambda, \quad (5.11)
\end{equation}

where \( \partial_{|w_k|^2} \) and \( \partial_{iz_k^\nu} \) are the Gelfand transforms of the operators

\[ F(w, z) \mapsto |w_k|^2 F(w, z) \quad \text{and} \quad F(w, z) \mapsto iz_k^\nu F(w, z), \]

respectively.

Now, since each

\[ \left( \frac{\alpha_k + n_k - 1}{\alpha_k} \right) \sim \alpha_k^{n_k-1} \sim 2^{j(n_k-1)}, \]

we see that

\[ D_G(\lambda) = \prod_{k=1}^{N} \left( \frac{\alpha_k + n_k - 1}{\alpha_k} \right) |\lambda_k|^{n_k} \sim \prod_{k=1}^{N} 2^{j(n_k-1)} 2^{L_k n_k} = 2^{L+j(n-N)}. \]

From well known properties of the Laguerre polynomials \( L_{\alpha}^{n-1} \) (see Appendix A), one observes that

\begin{equation}
\partial_{|w_k|^2} = -\frac{2}{|\lambda_k|} \left( \tau_k^{1}(\alpha_k + n_k - 1) \Delta_k^{q} + n_k \Delta_k \right) \quad (5.12)
\end{equation}

and

\begin{equation}
\partial_{iz_k^\nu} = \partial_{(k, \nu)} - \frac{\lambda_k^\nu}{2|\lambda_k|^2} \left( \tau_k^{1}(\alpha_k + n_k - 1) + \alpha_k \right) \Delta_k, \quad (5.13)
\end{equation}

where the operators \( \tau_k^q \) are defined by

\[ f(\alpha) \mapsto f(\alpha_1, \ldots, \alpha_k + q, \ldots, \alpha_N), \quad \alpha \in \mathbb{N}^N. \]
Note that, for each \( k = 1, \ldots, N \), if \( f \) has the same support as \( \tilde{m}_{j,l} \),

\[
\int_{\Pi_{k=1}^N s_k^{(0)}} \sum_{\alpha \in \mathbb{N}^N} |(1 + 2^{j+L_k} \partial_{|w_k|^2}) f(\alpha, \lambda)|^2 \, d\lambda = \int_{\Pi_{k=1}^N s_k^{(0)}} \sum_{\alpha \in \mathbb{N}^N} \left| \left( 1 - \frac{2^{j+L_k+1}}{|\lambda_k|} \left( \tau_k^1(\alpha_k + n_k - 1) \Delta_k^2 + n_k \Delta_k \right) \right) f(\alpha, \lambda) \right|^2 \, d\lambda
\]

\[
\lesssim \int_{\Pi_{k=1}^N s_k^{(0)}} \sum_{\alpha \in \mathbb{N}^N} |(1 + 2^{2j} |\Delta_k|^2 + 2^{j} |\Delta_k|) f(\alpha, \lambda)|^2 \, d\lambda
\]

\[
\leq \int_{\Pi_{k=1}^N s_k^{(0)}} \sum_{\alpha \in \mathbb{N}^N} |(1 + 2^{j} |\Delta_k|) f(\alpha, \lambda)|^2 \, d\lambda,
\]

where we have used the fact that, on the support of \( f \),

\[
|\alpha_k| \lesssim 2^j \quad \text{and} \quad |\lambda_k| \sim 2^{L_k}.
\]

Also, for each \( k \) and \( p = 1, \ldots, d_k \),

\[
\int_{\Pi_{k=1}^N s_k^{(0)}} \sum_{\alpha \in \mathbb{N}^N} \left| (1 + 2^{L_k} \partial_{iz_k^p}) f(\alpha, \lambda) \right|^2 \, d\lambda = \int_{\Pi_{k=1}^N s_k^{(0)}} \sum_{\alpha \in \mathbb{N}^N} \left| \left( 1 + 2^{L_k} \left( \frac{\lambda_k^p}{2|\lambda_k|} \right) (\tau_k^1(\alpha_k + n_k - 1) + \alpha_k) \Delta_k \right) f(\alpha, \lambda) \right|^2 \, d\lambda
\]

\[
\lesssim \int_{\Pi_{k=1}^N s_k^{(0)}} \sum_{\alpha \in \mathbb{N}^N} \left| (1 + 2^{L_k} |\partial_{iz_k^p}| + 2^{j} |\Delta_k|) f(\alpha, \lambda) \right|^2 \, d\lambda,
\]

where we have also used \( |\alpha_k| \lesssim 2^j \) and \( |\lambda_k| \sim 2^{L_k} \), as well as \( |\lambda_k^p| \sim 2^{L_k} \leq 2^{L_k} \) in the support of \( f \). These are the desired estimates on each of the operators

\[
1 + 2^{j+L_k} \partial_{|w_k|^2} \quad \text{and} \quad 1 + 2^{L_k} \partial_{iz_k^p}.
\]

In order to estimate (5.11), we note that we can rewrite (5.12) and (5.13) as

\[
\partial_{|w_k|^2} = -\frac{2}{|\lambda_k|} \left( \tau_k^1 \alpha_k \Delta_k^2 + ((n_k - 1) \tau_k^1 + 1) \Delta_k \right)
\]
\[ \partial_{iz_k^p} = \partial_{(k,p)} - \frac{\lambda_k^p}{2|\lambda_k|^2} ((\tau_k^1 + 1)\alpha_k \Delta_k + (n_k - 1)(\tau_k^1 - 1)). \]

And thus, using the commutators

\[ \Delta_k \alpha_k = \alpha_k \Delta_k + \tau_k^{-1}, \quad \alpha_k \tau_k^q = \tau_k^q \alpha_k - q \tau_k^q, \]

together with \( \Delta_k = 1 - \tau_k^{-1} \), we obtain the expansion

\[
(1 + 2^{j+L_k} \partial_{|w_k|^2})^{n_k} \prod_{p=1}^{d_k} (1 + 2^{l_k} \partial_{iz_k^p}) = \\
\sum_{0 \leq \mu \leq n_k} \mathcal{A}(n_k, d_k, \mu, \nu, \varepsilon, \delta) \frac{2^{\mu(j+L_k)}}{|\lambda_k|^{\mu+2|\delta|}} \prod_{1 \leq p, q \leq d_k} (2^{l_k} \partial_{(k,p)})^{\varepsilon_p} (2^{l_k} \lambda_k^q) \delta_q \alpha_k^\nu \Delta_k^{\mu+\mu}, \quad (5.14)
\]

where the sum is taken over all \( d_k \)-multiindexes \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_{d_k}), \delta = (\delta_1, \ldots, \delta_{d_k}) \) with \( 0 \leq \varepsilon_p, \delta_q \leq 1, \ p, q = 1, \ldots, d_k, \ |\delta| = \delta_1 + \ldots + \delta_{d_k} \), and the \( \mathcal{A}(n_k, d_k, \mu, \nu, \varepsilon, \delta) \) are polynomials in \( \tau_k \) and \( \tau_k^{-1} \) (see Appendix B). We have then

\[
\int \prod_{k=1}^{N} \sum_{\alpha \in \mathbb{N}^N} \left| \prod_{k=1}^{N} (1 + 2^{j+L_k} \partial_{|w_k|^2})^{n_k} \prod_{p=1}^{d_k} (1 + 2^{l_k} \partial_{iz_k^p}) \bar{m}_{j,l}(\alpha, \lambda) \right|^2 D_G(\lambda) d\lambda \\
\lesssim 2^{L+j(n-N)} \int \prod_{k=1}^{N} \sum_{\alpha \in \mathbb{N}^N} \left| \prod_{k=1}^{N} \left( \sum_{0 \leq \mu \leq n_k} \sum_{0 \leq \varepsilon_p, \delta_q \leq 1} \mathcal{A}(n_k, d_k, \mu, \nu, \varepsilon, \delta) \frac{2^{\mu(j+L_k)}}{|\lambda_k|^{\mu+2|\delta|}} \right) \prod_{1 \leq p, q \leq d_k} (2^{l_k} \partial_{(k,p)})^{\varepsilon_p} (2^{l_k} \lambda_k^q) \delta_q \alpha_k^\nu \Delta_k^{\mu+\mu} \bar{m}_{j,l}(\alpha, \lambda) \right|^2 d\lambda \\
\lesssim 2^{L+j(n-N)} \int \prod_{k=1}^{N} \sum_{\alpha \in \mathbb{N}^N} \left| \prod_{k=1}^{N} \left( \sum_{0 \leq \mu \leq n_k} \sum_{0 \leq \varepsilon_p \leq 1} \prod_{1 \leq p \leq d_k} (2^{l_k} \partial_{(k,p)})^{\varepsilon_p} (2^{l_k} |\Delta_k|)^{\mu+\varepsilon} \bar{m}_{j,l}(\alpha, \lambda) \right|^2 d\lambda,
\]

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where again we have used the estimates
\[
|\alpha_k| \lesssim 2^j \quad \text{and} \quad |\lambda_k| \sim 2^{L_k},
\]
as well as \(|\lambda_k^p| \sim 2^{L_k} \leq 2^{L_k}\) in the support of \(\tilde{m}_{j,l}\). Finally, from the elementary identities
\[
(1+r)^n = \sum_{k=0}^{n} \binom{n}{k} r^k, \quad \prod_{p=1}^{d} (1+r+s_p) = \sum_{0 \leq k \leq d} C(d,k,\varepsilon_1,\ldots,\varepsilon_d) r^k s_1^{\varepsilon_1} \cdots s_d^{\varepsilon_d},
\]
we obtain
\[
\int_{\prod_{k=1}^{N} S_k \setminus \{0\}} \sum_{\alpha \in \mathbb{N}^N} \left| \prod_{k=1}^{N} (1 + 2^{L_k} \partial_{w_k} |z|^2)^{n_k} \prod_{p=1}^{d_k} (1 + 2^{L_p} \partial_{z_{k,p}}) \tilde{m}_{j,l}(\alpha, \lambda) \right|^2 D_G(\lambda) d\lambda \\
\lesssim 2^{L_j(n-N)} \int_{\prod_{k=1}^{N} S_k \setminus \{0\}} \sum_{\alpha \in \mathbb{N}^N} \left| \prod_{k=1}^{N} (1 + 2^j |\Delta_k|)^{2n_k} \prod_{p=1}^{d_k} (1 + 2^{L_p} |\partial_{(k,p)}| + 2^{j} |\Delta_k|) \tilde{m}_{j,l}(\alpha, \lambda) \right|^2 d\lambda.
\]
The integral finishes the proof of Lemma 5.4.
Chapter 6

Conclusions

Our main results, Corollaries 2.5, 3.7, 3.8, and 5.2 provide the following facts:

1. A multiparameter regularity condition of order $D/2$, where $D$ is the real dimension of the product of $H$-type groups $G = G_1 \times \cdots \times G_n$, implies the boundedness of Marcinkiewicz operators of the form

$$m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iZ_1^1, \ldots, -iZ_n^d)$$

on $L^p(G)$.

2. A regularity condition of order $3n/2$ implies the boundedness of Marcinkiewicz operators of the form

$$m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT_1, \ldots, -iT_d)$$

on $L^p((\mathbb{H}_1)^n/K)$, for a central $(n - d)$-dimensional subgroup $K$ of $(\mathbb{H}_1)^n$.

Fact 1 extends the results of Müller, Ricci and Stein [MRS96] where they studied operators of the form $m(\mathcal{L}, -iT)$, to products of $H$-type groups. As in they work, we have made use of the explicit formulae of the Fourier analysis on the Heisenberg
group (cf. Chapter 1) and on an $H$-type group (cf. Chapter 4).

We must note that, in Chapters 1 and 2, we have restricted ourselves to the study of Marcinkiewicz multipliers on products of 3-dimensional Heisenberg groups only, in order to move on to quotients of the form $(\mathbb{H}_1)^n/K$. However, one can extend Theorem 2.4 to an analogous result on operators of the form

$$m(\mathcal{L}_1^1, \ldots, \mathcal{L}_1^2, \ldots, \mathcal{L}_N^1, \ldots, \mathcal{L}_N^N, -iT_1, \ldots, -iT_N)$$

on products $\mathbb{H}_{n_1} \times \cdots \times \mathbb{H}_{n_N}$, where $\mathcal{L}_j^1, \ldots, \mathcal{L}_j^{n_j}$ are the partial sublaplacians on each group $\mathbb{H}_{n_j}$. In [Ven00b] Veneruso studies such operators on a single Heisenberg group, and one can use the techniques of Chapter 2 to extend Veneruso’s results to products.

Fact 2 further extends the theory of Marcinkiewicz multipliers, now to central quotients $G_A = (\mathbb{H}_1)^n/K$ of products of Heisenberg groups. Though we have not proved that a regularity condition of order $D/2$ is sufficient to guarantee boundedness on $L^p$, we see that, if the dimension $d$ of its center satisfies $d > n/2$, then the regularity condition $3n/2$ is less than $Q/2$, where $Q = 2n + 2d$ is the homogenous dimension of $G_A$.

Besides the explicit Fourier theory on $G_A$, we have as well used a method of “lifting”, adding variables to the function $m$ to reduce the result to a product case. Such approach has been successful in other contexts, as the study of vector fields, in order to make use of homogeneity (see, for example, [CNSW99]).

Fact 2 also provides another set of examples of homogeneous groups $G$ where a regularity condition of order less than $Q/2$ implies the boundedness of an spectral multiplier operator. The following question is still open:

For which homogeneous groups $G$ does a regularity condition of order $D/2$ imply the boundedness of an spectral multiplier operator?
Besides the results mentioned here, it is known, as stated in [Heb93], that there exist other groups $G$ where a regularity condition of order less than $Q/2$ is enough for the boundedness of an spectral multiplier. However, the above question is still open in the general case.

Out of the results of this thesis, one can then also ask the following (widely open) questions:

1. For which homogeneous groups $G$, and vector fields $X_1, \ldots, X_k$ on $G$, can one study the boundedness of operators $m(X_1, \ldots, X_k)$ from the regularity of $m$?

2. What conditions must the group $G$ and normal subgroups $N$ of $G$ satisfy in order to conclude statements for the regularity of multipliers on $G/N$ out of multiplier theorems for $G$?
Appendix A

Gelfand transforms of multiplication operators

In this appendix we prove the formulas (5.12) and (5.13) on the product group $G$. One notices that these proofs also apply to the formulas (2.14) for the operators $\partial_{|z_k|^2}$ and $\partial_{\delta_k}$ on $(\mathbb{H}_1)^n$, as the latter are special cases when each $G_j = \mathbb{H}_1$ in the product $G$.

We start by recalling the identities for the Laguerre polynomials

$$
(\alpha + 1)L_{\alpha+1}^n(x) = (2\alpha + n + 1 - x)L_{\alpha}^n(x) - (\alpha + n)L_{\alpha-1}^n, \quad (A.1)
$$

which provide a recursion formula for $L_{\alpha}^n$, and

$$
x \frac{d}{dx} L_{\alpha}^n(x) = \alpha L_{\alpha}^n(x) - (\alpha + n)L_{\alpha-1}^n(x). \quad (A.2)
$$

These may be found, for instance, in [Tha93].

We proceed to calculate the Gelfand transform of the operator

$$
F(w, z) \mapsto |w_k|^2 F(w, z).
$$
From the inversion formula (4.10),

\[
|w_k|^2 F(w, z) = C_g \int_{\Pi_j=1 \cup \{0\}} \sum_{\alpha \in \mathbb{N}^N} \hat{F}(\alpha, \lambda) |w_k|^2 \phi_{\alpha, \lambda}(w, z) D_G(\lambda) d\lambda_1 \ldots d\lambda_N. \tag{A.3}
\]

By (4.8) and (4.2),

\[
|w_k|^2 \phi_{\alpha, \lambda}(w, z) = \prod_{j \neq k} \phi_{\alpha_j, \lambda_j}(w, z) \times \left( \frac{\alpha_k + n_k - 1}{\alpha_k} \right)^{-1} e^{i\lambda_k(z_k)} e^{-|\lambda_k||w_k|^2} |w_k|^2 L_{\alpha_k}^{n_k-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right),
\]

and, by (A.1),

\[
|w_k|^2 L_{\alpha_k}^{n_k-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) = \frac{2}{|\lambda_k|} \left( (2\alpha_k + n_k) L_{\alpha_k}^{n_k-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) - (\alpha_k + 1) L_{\alpha_k+1}^{n_k-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) - (\alpha_k + n_k - 1) L_{\alpha_k-1}^{n_k-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) \right).
\]

Therefore, by changing the summation indices in (A.3) we obtain

\[
|w_k|^2 F(w, z) = C_g \int_{\Pi_j=1 \cup \{0\}} \sum_{\alpha \in \mathbb{N}^N} \partial_{|w_k|^2} \hat{F}(\alpha, \lambda) \phi_{\alpha, \lambda}(w, z) D_G(\lambda) d\lambda_1 \ldots d\lambda_N.
\]

where

\[
\partial_{|w_k|^2} \hat{F}(\alpha, \lambda) = \frac{2}{|\lambda_k|} \left( (2\alpha_k + n_k) \hat{F}(\alpha, \lambda) - \alpha_k \tau_k^{-1} \hat{F}(\alpha, \lambda) - (\alpha_k + n_k) \tau_k^{-1} \hat{F}(\alpha, \lambda) \right).
\]

Thus

\[
\partial_{|w_k|^2} = \frac{2}{|\lambda_k|} \left( (2\alpha_k + n_k) - \alpha_k \tau_k^{-1} - (\alpha_k + n_k) \tau_k^{-1} \right) = \frac{2}{|\lambda_k|} \left( \alpha_k (1 - \tau_k^{-1}) - (\alpha_k + n_k) (\tau_k^{-1} - 1) \right).
\]

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Now, from the definition of $\Delta_k$ and $\tau_k^p$, we have the relations

$$\Delta_k = 1 - \tau_k^{-1}, \quad \alpha_k \tau_k^1 = \tau_k^1 \alpha_k - \tau_k^1, \quad \text{and} \quad \Delta_k \alpha_k = \alpha_k \Delta_k + \tau_k^{-1}.$$ 

Therefore

$$\alpha_k (1 - \tau_k^{-1}) - (\alpha_k + n_k) (\tau_k^1 - 1) = \alpha_k \Delta_k - (\alpha_k + n_k) \tau_k^1 \Delta_k$$

$$= \alpha_k \Delta_k - (\tau_k^1 (\alpha_k + n_k) - \tau_k^1) \Delta_k$$

$$= -\tau_k^1 (\Delta_k \alpha_k + (n_k - 1)) \Delta_k$$

$$= -\tau_k^1 (\alpha_k \Delta_k + \tau_k^{-1} (n_k - 1)) \Delta_k$$

$$= -\tau_k^1 (\alpha_k \Delta_k + (n_k - 1) \Delta_k + \tau_k^{-1} n_k) \Delta_k$$

$$= -\tau_k^1 (\alpha + n_k - 1) \Delta_k^2 - n_k \Delta_k,$$

and thus

$$\partial_{|w_k|^2} = -\frac{2}{|\lambda_k|} (\tau_k^1 (\alpha + n_k - 1) \Delta_k^2 + n_k \Delta_k),$$

which is equation (5.12).

For the operator $F(w, z) \mapsto iz_k^p F(w, z)$, we see that

$$iz_k^p F(w, z) = C_g \int_{\Pi_j^N j_j \neq \emptyset} \sum_{\alpha \in \mathbb{N}^N} \hat{F}(\alpha, \lambda) iz_k^p \phi_{G, \alpha, \lambda}^G (w, z) D_G (\lambda) d\lambda_1 \ldots d\lambda_N,$$  \hspace{1cm} (A.4)

and

$$iz_k^p \phi_{G, \alpha, \lambda}^G (w, z) = \prod_{j \neq k} \phi_{\alpha_j, \lambda_j}^G (w, z) \times \left( \frac{\alpha_k + n_k - 1}{\alpha_k} \right)^{-1} iz_k^p e^{i\lambda_k (z_k)} e^{-\frac{|\lambda_k||w_k|^2}{4}} L_{\alpha_k}^{-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right).$$

We observe that

$$iz_k^p e^{i\lambda_k (z_k)} = \frac{\partial}{\partial \lambda_k^p} e^{i\lambda_k (z_k)},$$

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Therefore, by modifying the summation indices in (A.5) we obtain

\[ i z_k^p F(w, z) = \]

\[ - C_\mathfrak{g} \int_{\prod_{j=1}^{N} \{0\} \setminus \{0\}} \prod_{\alpha \in \mathbb{N}^N \setminus \{0\}} \left\{ \frac{\alpha_j \lambda_j}{\alpha_j} \left( \frac{\alpha_j + n_j - 1}{\alpha_j} \right)^{-1} |\lambda_j|^{n_j} \right\} e^{i \lambda_k(z_k)} \times \]

\[ \left( \alpha_k + n_k - 1 \right)^{-1} \frac{\partial}{\partial \lambda_k} \left\{ \hat{F}(\alpha, \lambda) L_{\alpha_k}^{-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) e^{-\frac{\lambda_k ||w_k||^2}{4}} |\lambda_k|^{n_k} \right\} \]

\[ d\lambda_1 \ldots d\lambda_N. \quad (A.5) \]

Now

\[ \frac{\partial}{\partial \lambda_k} \left\{ \hat{F}(\alpha, \lambda) L_{\alpha_k}^{-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) e^{-\frac{\lambda_k ||w_k||^2}{4}} |\lambda_k|^{n_k} \right\} \]

\[ = \partial_{(k,p)} \hat{F}(\alpha, \lambda) L_{\alpha_k}^{-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) e^{-\frac{\lambda_k ||w_k||^2}{4}} |\lambda_k|^{n_k} \]

\[ + \hat{F}(\alpha, \lambda) e^{-\frac{\lambda_k ||w_k||^2}{4}} |\lambda_k|^{n_k} \times \left\{ \frac{1}{2} |w_k|^2 \frac{\lambda_k^p}{|\lambda_k|} (L_{\alpha_k}^{-1})' \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) \right\} \]

\[ - \frac{1}{4} |w_k|^2 \frac{\lambda_k^p}{|\lambda_k|} L_{\alpha_k}^{-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) + n_k \frac{\lambda_k^p}{|\lambda_k|} L_{\alpha_k}^{-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) \}, \]

and, by (A.1) and (A.2),

\[ \frac{1}{2} |w_k|^2 \frac{\lambda_k^p}{|\lambda_k|} (L_{\alpha_k}^{-1})' \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) - \frac{1}{4} |w_k|^2 \frac{\lambda_k^p}{|\lambda_k|} L_{\alpha_k}^{-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) \]

\[ + n_k \frac{\lambda_k^p}{|\lambda_k|} L_{\alpha_k}^{-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) = \frac{\lambda_k^p}{2|\lambda_k|^2} \left\{ n_k L_{\alpha_k}^{-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) \right\} \]

\[ - (\alpha_k + n_k - 1) L_{\alpha_k-1}^{-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) \]

\[ + (\alpha_k + 1) L_{\alpha_k+1}^{-1} \left( \frac{1}{2} |\lambda_k||w_k|^2 \right) \}. \]

Therefore, by modifying the summation indices in (A.5) we obtain

\[ i z_k^p F(w, z) = C_\mathfrak{g} \int_{\prod_{j=1}^{N} \{0\} \setminus \{0\}} \sum_{\alpha \in \mathbb{N}^N} \partial_{i z_k} \hat{F}(\alpha, \lambda) \overline{\phi_{\alpha, \lambda}(w, z)} D_G(\lambda) d\lambda_1 \ldots d\lambda_N, \]
where
\[
\partial_{iz_k^p} \hat{F}(\alpha, \lambda) = \partial_{(k,p)} \hat{F}(\alpha, \lambda) + \frac{\lambda_k^p}{2|\lambda_k|^2} (n_k \hat{F}(\alpha, \lambda) - (\alpha_k + n_k) \tau_k^1 \hat{F}(\alpha, \lambda) + \alpha_k \tau_k^{-1} \hat{F}(\alpha, \lambda)).
\]

Therefore
\[
\partial_{iz_k^p} = \partial_{(k,p)} + \frac{\lambda_k^p}{2|\lambda_k|^2} (n_k - (\alpha_k + n_k) \tau_k^1 + \alpha_k \tau_k^{-1})
= \partial_{(k,p)} - \frac{\lambda_k^p}{2|\lambda_k|^2} ((\alpha_k + n_k) \tau_k^1 + \alpha_k) \Delta_k
= \partial_{(k,p)} - \frac{\lambda_k^p}{2|\lambda_k|^2} (\tau_k^1 (\alpha_k + n_k - 1) + \alpha_k) \Delta_k.
\]

We have thus obtained (5.13).
Appendix B

Expansions of difference operators

In this appendix we prove the expansion (5.14), used in the proof of Lemma 5.4. As noted, such expansion follows from the commutator relations

\[ \Delta_k \alpha_k = \alpha_k \Delta_k + \tau_k^{-1} \quad \text{and} \quad \alpha_k \tau_k^p = \tau_k^p \alpha_k - p \tau_k^p. \quad (B.1) \]

We recall that \( \partial |w_k|^2 = -\frac{2}{|\lambda_k|} (\tau_k^1 \alpha_k \Delta_k^2 + ((n_k - 1) \tau_k^1 + 1) \Delta_k) \), and we prove first

\[ (\partial |w_k|^2)^\mu = \left( -\frac{2}{|\lambda_k|} (\tau_k^1 \alpha_k \Delta_k^2 + ((n_k - 1) \tau_k^1 + 1) \Delta_k) \right)^\mu = \frac{1}{|\lambda_k|^{\mu}} \sum_{\nu=0}^{\mu} A_{\mu}^\nu \alpha_k^\nu \Delta_k^{\nu+\mu}, \quad (B.2) \]

where \( A_{\mu}^\nu \) is a polynomial in \( \tau_k^1 \) and \( \tau_k^{-1} \), i.e. a finite linear combination of operators \( \tau_k^p, p \in \mathbb{Z} \). For ease of notation we shall remove the subindex \( k \), and write simply \( \tau \) for \( \tau^1 \).

We proceed by induction. Note that (B.2) is true with \( \mu = 1 \), with \( A_1^1 = -2 \tau \) and \( A_0^1 = -2((n - 1) \tau + 1) \). We assume (B.2) is true for \( 1, \ldots, \mu \), and thus we have to expand

\[ \sum_{\nu=0}^{\mu} (\tau \alpha \Delta^2 + ((n - 1) \tau + 1) \Delta) A_{\mu}^\nu \alpha^\nu \Delta^{\nu+\mu}. \quad (B.3) \]
This involves dealing with

\[ \alpha \Delta^2 \tau^p \alpha^\nu \Delta^{\nu+\mu} \quad \text{and} \quad \Delta \tau^p \alpha^\nu \Delta^{\nu+\mu}, \]

as (B.3) is a linear combination of such operators. As \( \Delta = 1 - \tau^{-1} \), \( \Delta \) and \( \tau \) commute, so we have

\begin{align*}
\alpha \Delta^2 \tau^p \alpha^\nu \Delta^{\nu+\mu} &= \alpha \tau^p \Delta^2 \alpha^\nu \Delta^{\nu+\mu} = (\tau^p \alpha - p \tau^p) \Delta^2 \alpha^\nu \Delta^{\nu+\mu} \\
&= \tau^p \alpha \Delta^2 \alpha^\nu \Delta^{\nu+\mu} - p \tau^p \Delta^2 \alpha^\nu \Delta^{\nu+\mu},
\end{align*}

and

\[ \Delta \tau^p \alpha^\nu \Delta^{\nu+\mu} = \tau^p \Delta \alpha^\nu \Delta^{\nu+\mu}. \]

We note that \( p \tau^p \Delta^2 \alpha^\nu \Delta^{\nu+\mu} = p \tau^p (1 - \tau^{-1}) \Delta \alpha^\nu \Delta^{\nu+\mu} \), and \( p \tau^p (1 - \tau^{-1}) = p \tau^p - p \tau^{p-1} \) is a polynomial in \( \tau \), so we need to prove that \( \alpha \Delta^2 \alpha^\nu \Delta^{\nu+\mu} \) and \( \Delta \alpha^\nu \Delta^{\nu+\mu} \) are of the form

\begin{align*}
\sum_{r=0}^{\nu+1} P(\nu, r) \alpha^r \Delta^{r+\mu+1} \quad \text{and} \quad \sum_{r=0}^\nu Q(\nu, r) \alpha^r \Delta^{r+\mu+1},
\end{align*}

respectively, where \( P(\nu, r) \) and \( Q(\nu, r) \) are polynomials in \( \tau \) and \( \tau^{-1} \).

We proceed in steps. We first prove the following.

**Step 1.** For \( k, n \in \mathbb{N}, p \in \mathbb{Z}, \)

\[ \alpha^k \tau^p \Delta^{k+n} = \sum_{r=0}^k D(k, p, r) \alpha^r \Delta^{r+n}, \quad \text{(B.4)} \]

where the \( D(k, p, r) \) are polynomials in \( \tau \) and \( \tau^{-1} \).

**Proof.** We observe that (B.4) is trivial for \( k = 0 \). Assuming it is true for \( 0, 1, \ldots, k, \)
we see that
\[
\alpha^{k+1} \tau^p \Delta^{k+1+n} = \alpha \left( \sum_{r=0}^{k} D(k, p, r) \alpha^r \Delta^{r+n} \right) \Delta \\
= \sum_{r=0}^{k} \left( D(k, p, r) \alpha + \tilde{D}(k, p, r) \right) \alpha^r \Delta^{r+n+1} \\
= \sum_{r=0}^{k} D(k, p, r) \alpha^{r+1} \Delta^{r+1+n+1} + \sum_{r=0}^{k} \tilde{D}(k, p, r) \alpha^r \Delta^{r+n+1},
\]
where the \( \tilde{D}(k, p, r) \) are also polynomials in \( \tau \) and \( \tau^{-1} \). We have used (B.1). The first sum is already in the form of (B.4). For the second, we note that
\[
\alpha^r \Delta^{r+n+1} = \alpha^r (1 - \tau^{-1}) \Delta^{r+n} = \alpha^r \Delta^{r+n} - \alpha^r \tau^{-1} \Delta^{r+n}.
\]
From the induction hypothesis, as \( 0 \leq r \leq k \), \( \alpha^r \tau^{-1} \Delta^{r+n} = \sum_{s=0}^{r} D(r, -1, s) \alpha^s \Delta^{s+n} \), and thus
\[
\sum_{r=0}^{k} \tilde{D}(k, p, r) \alpha^r \Delta^{r+n+1} = \sum_{r=0}^{k} \tilde{D}(k, p, r) \alpha^r \Delta^{r+n} - \sum_{r=0}^{k} \tilde{D}(k, p, r) \alpha^r \tau^{-1} \Delta^{r+n} \\
= \sum_{r=0}^{k} \tilde{D}(k, p, r) \alpha^r \Delta^{r+n} \\
- \sum_{r=0}^{k} \tilde{D}(k, p, r) \sum_{s=0}^{r} D(r, -1, s) \alpha^s \Delta^{s+n},
\]
which is also of the desired form.

\[\square\]

**Step 2.** For \( k, j, n \in \mathbb{N}, p \in \mathbb{Z} \),
\[
\alpha^k \tau^p \alpha^j \Delta^{k+j+n} = \sum_{r=0}^{k+j} C(k, j, p, r) \alpha^r \Delta^{r+n}, \quad (B.5)
\]
where the \( C(k, j, p, r) \) are polynomials in \( \tau \) and \( \tau^{-1} \).

**Proof.** As in Step 1, we see that (B.5) is trivial for \( k = 0 \). Assuming it is true for
\[ \alpha^{k+1} r^p \alpha^j \Delta^{k+1+j+n} = \alpha \left( \sum_{r=0}^{k+j} C(k, j, p, r) \alpha^r \Delta^{r+n} \right) \Delta \]

\[ = \sum_{r=0}^{k+j} \left( C(k, j, p, r) \alpha + \tilde{C}(k, j, p, r) \right) \alpha^r \Delta^{r+n+1} \]

\[ = \sum_{r=0}^{k+j} C(k, j, p, r) \alpha^{r+1} \Delta^{r+1+n} + \sum_{r=0}^{k+j} \tilde{C}(k, j, p, r) \alpha^r (1 - \tau^{-1}) \Delta^{r+n} \]

\[ = \sum_{r=0}^{k+j} C(k, j, p, r) \alpha^{r+1} \Delta^{r+1+n} + \sum_{r=0}^{k+j} \tilde{C}(k, j, p, r) \alpha^r \Delta^{r+n} \]

\[ - \sum_{r=0}^{k+j} \tilde{C}(k, j, p, r) \sum_{s=0}^r D(r, -1, s) \alpha^s \Delta^{s+n}, \]

where we have used Step 1 in the last equality. \hfill \square

We observe that we can extend Step 2 to any polynomial \( P = \sum a_p \tau^p \) in \( \tau \) and \( \tau^{-1} \), i.e.

\[ \alpha^k P \alpha^j \Delta^{k+j+n} = \sum_{r=0}^{k+j} C_P(k, j, r) \alpha^r \Delta^{r+n}, \] (B.6)

where we simply have \( C_P(k, j, r) = \sum a_p C(k, j, p, r) \).

We now first prove

\[ \Delta \alpha^\nu \Delta^{\nu+\mu} = \sum_{r=0}^\nu Q(\nu, r) \alpha^r \Delta^{r+\mu+1}. \] (B.7)

This identity is trivial for \( \nu = 0 \). Assuming (B.7) is true for \( 0, 1, \ldots, \nu \), we have

\[ \Delta \alpha^{\nu+1} \Delta^{\nu+1+\mu} = (\alpha \Delta + \tau^{-1}) \alpha^{\nu} \Delta^{\nu+1+\mu} = \alpha \Delta \alpha^{\nu} \Delta^{\nu+\mu+1} + \tau^{-1} \alpha^{\nu} \Delta^{\nu+\mu+1}. \]

The last term is of the desired form. For the former, we use the induction hypothesis.
and thus

$$\alpha \Delta \alpha^\nu \Delta^{\nu+\mu+1} = \alpha \left( \sum_{r=0}^\nu Q(\nu, r) \alpha^r \Delta^{r+\mu+1} \right) \Delta$$

$$= \sum_{r=0}^\nu \left( Q(\nu, r) \alpha + \tilde{Q}(\nu, r) \right) \alpha^r \Delta^{r+\mu+2}$$

$$= \sum_{r=0}^\nu Q(\nu, r) \alpha^{r+1} \Delta^{r+1+\mu+1} + \sum_{r=0}^\nu \tilde{Q}(\nu, r) \alpha^r (1 - \tau^{-1}) \Delta^{r+\mu+1}$$

$$= \sum_{r=0}^\nu Q(\nu, r) \alpha^{r+1} \Delta^{r+1+\mu+1} + \sum_{r=0}^\nu \tilde{Q}(\nu, r) \alpha^r \Delta^{r+\mu+1}$$

$$- \sum_{r=0}^\nu \tilde{Q}(\nu, r) \sum_{s=0}^r D(r, -1, s) \alpha^s \Delta^{s+\mu+1},$$

which is of the form (B.7).

We now turn to prove

$$\alpha \Delta^2 \alpha^\nu \Delta^{\nu+\mu+1} = \sum_{r=0}^{\nu+1} P(\nu, r) \alpha^r \Delta^{r+\mu+1}. \quad (B.8)$$

(B.8) is trivial for $\nu = 0$. Assuming it is true for $0, 1, \ldots, \nu$, we take

$$\alpha \Delta^2 \alpha^{\nu+1} \Delta^{\nu+1+\mu} = \alpha \Delta(\Delta \alpha) \alpha^\nu \Delta^{\nu+1+\mu} = \alpha \Delta(\alpha \Delta + \tau^{-1}) \alpha^\nu \Delta^{\nu+1+\mu}$$

$$= \alpha (\alpha \Delta + \tau^{-1}) \Delta \alpha^\nu \Delta^{\nu+1+\mu} + \alpha \Delta \tau^{-1} \alpha^\nu \Delta^{\nu+1+\mu}$$

$$= \alpha^2 \Delta^2 \alpha^\nu \Delta^{\nu+1+\mu} + 2\alpha \tau^{-1} \Delta \alpha^\nu \Delta^{\nu+1+\mu},$$

where we have used the fact that $\tau^{-1}$ and $\Delta$ commute. Now, by the induction
hypothesis, 

\[ \alpha^2 \Delta^2 \alpha^r \Delta^{\nu+1+\mu} = \alpha(\alpha^2 \alpha^r \Delta^{\nu+\mu})\Delta = \alpha \left( \sum_{r=0}^{\nu+1} P(\nu, r) \alpha^r \Delta^{r+\mu+1} \right) \Delta \]

\[ = \sum_{r=0}^{\nu+1} \left( P(\nu, r) \alpha + \tilde{P}(\nu, r) \right) \alpha^r \Delta^{r+\mu+2} \]

\[ = \sum_{r=0}^{\nu+1} P(\nu, r) \alpha^{r+1} \Delta^{r+1+\mu+1} + \sum_{r=0}^{\nu+1} \tilde{P}(\nu, r) \alpha^r (1 - \tau^{-1}) \Delta^{r+\mu+1} \]

\[ = \sum_{r=0}^{\nu+1} P(\nu, r) \alpha^{r+1} \Delta^{r+1+\mu+1} + \tilde{P}(\nu, r) \alpha^r \Delta^{r+\mu+1} \]

\[ - \sum_{r=0}^{\nu+1} \tilde{P}(\nu, r) \sum_{s=0}^{r} D(r, -1, s) \alpha^s \Delta^{s+\mu+1}, \]

which is in the form (B.8). We have used Step 1 in the last identity. Using (B.7),

\[ \alpha^{\nu} \Delta^{\nu} \Delta^{\nu+1+\mu} = (\tau^{-1} \alpha + \tau^{-1}) \Delta^{\nu} \Delta^{\nu+1+\mu} = \tau^{-1} \alpha \Delta^{\nu} \Delta^{\nu+1+\mu} + \tau^{-1} \Delta \alpha^{\nu} \Delta^{\nu+1+\mu} \]

\[ = \tau^{-1} \alpha \left( \sum_{r=0}^{\nu} Q(\nu, r) \alpha^r \Delta^{r+\mu+1} \right) \Delta + \tau^{-1} \Delta \alpha^{\nu} \Delta^{\nu+1+\mu} \]

\[ = \sum_{r=0}^{\nu} \tau^{-1} Q(\nu, r) \alpha^{r+1} \Delta^{r+1+\mu+1} + \sum_{r=0}^{\nu} \tau^{-1} \tilde{Q}(\nu, r) \alpha^r (1 - \tau^{-1}) \Delta^{r+\mu+1} \]

\[ + \tau^{-1} \Delta \alpha^{\nu} \Delta^{\nu+1+\mu} \]

\[ = \sum_{r=0}^{\nu} \tau^{-1} Q(\nu, r) \alpha^{r+1} \Delta^{r+1+\mu+1} + \sum_{r=0}^{\nu} \tau^{-1} \tilde{Q}(\nu, r) \alpha^r \Delta^{r+\mu+1} \]

\[ - \sum_{r=0}^{\nu} \tau^{-1} \tilde{Q}(\nu, r) \sum_{s=0}^{r} D(r, -1, s) \alpha^s \Delta^{s+\mu+1} + \tau^{-1} \Delta \alpha^{\nu} \Delta^{\nu+1+\mu}, \]

which is of the desired form since \( \tau^{-1} \Delta = \tau^{-1} - \tau^{-2} \).

As noted before, (B.7) and (B.8) imply that (B.3) is of the form

\[ \sum_{\nu=0}^{\mu+1} A_{\nu}^{\mu+1} \alpha^\nu \Delta^{\nu+\mu+1}, \]

and we conclude (B.2).
We therefore obtain

\[(1 + 2^j L |w|^2)^n = \sum_{\mu=0}^{n} \binom{n}{\mu} (2^j L |w|^2)^\mu = \sum_{0 \leq \nu \leq \mu \leq n} A(n, \mu, \nu) \frac{2^\mu(j+L)}{\lambda^\mu} \alpha^\nu \Delta^{\nu + \mu}, \quad (B.9)\]

where the \(A(n, \mu, \nu)\) are polynomials in \(\tau\) and \(\tau^{-1}\).

We now proceed to expand the product \(\prod_{p=1}^{d_k} (1 + 2^p \partial_{i z^p})\) for each \(k = 1, \ldots, N\). We first recall that

\[\partial_{i z^p} = \partial_{(k,p)} - \frac{\lambda_p^k}{2|\lambda_k|^2} ((\tau_k + 1)\alpha_k \Delta_k + (n_k - 1)(\tau_k - 1)),\]

where \(\lambda_p^k\) is the \(p\)-th coordinate of \(\lambda_k\) and \(\partial_{(k,p)}\) is the partial derivative with respect to \(\lambda_p^k\). As before, for simplicity, we remove the subindex \(k\), and write simply \(\partial_p\) for \(\partial_{(k,p)}\).

We’ll prove, inductively, for \(\sigma = 1, 2, \ldots, d(= d_k)\), that

\[\prod_{p=1}^{\sigma} (1 + 2^p \partial_{i z^p}) = \sum_{\varepsilon, \delta} B(\sigma, \varepsilon, \delta, \nu) \prod_{0 \leq p, q \leq \sigma} (2^p \partial_p)^{\varepsilon_p} (2^q \lambda^q)^{\delta_q} |\lambda|^{-2|\delta|} \alpha^{\nu} \Delta^{\nu}, \quad (B.10)\]

where the sum is taken over all \(\sigma\)-multiindexes \(\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\sigma), \delta = (\delta_1, \ldots, \delta_\sigma)\) with

\[0 \leq \varepsilon_p, \delta_q \leq 1, \quad p, q = 1, \ldots, \sigma; \quad \text{and} \quad |\delta| = \delta_1 + \ldots + \delta_\sigma,\]

and the \(B(\sigma, \varepsilon, \delta, \nu)\) are polynomials in \(\tau\) and \(\tau^{-1}\). For \(\sigma = 1\) we have

\[
1 + 2^i \partial_{i z^1} = 1 + 2^i \left( \partial_1 - \frac{\lambda_1}{2|\lambda|^2} ((\tau + 1)\alpha \Delta + (n - 1)(\tau - 1)) \right)
\]
\[
= 1 + 2^i \partial_1 - \frac{2^i \lambda_1}{2|\lambda|^2} (n - 1)(\tau - 1) - \frac{2^i \lambda_1}{2|\lambda|^2} (\tau + 1)\alpha \Delta
\]
\[
= B(1, 0, 0, 0) + B(1, 1, 0, 0)(2^i \partial_1) + B(1, 0, 1, 0)(2^i \lambda_1)|\lambda|^{-2}
\]
\[
+ B(1, 0, 1, 1)(2^i \lambda_1)|\lambda|^{-2} \alpha \Delta,
\]

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with \( B(1,0,0,0) = B(1,1,0,0) = 1, \)

\[
B(1,0,1,0) = -\frac{1}{2}(n-1)(\tau - 1), \quad \text{and} \quad B(1,0,1,1) = -\frac{1}{2}(\tau + 1).
\]

We assume (B.10) is true for \( \sigma \). Then

\[
\prod_{p=1}^{\sigma+1} (1 + 2^p \partial_{iz^p}) = (1 + 2^{\sigma+1} \partial_{iz^{\sigma+1}}) \prod_{p=1}^{\sigma} (1 + 2^p \partial_{iz^p}) \\
= (B(1,0,0,0) + B(1,1,0,0)(2^1 \partial_1) + B(1,0,1,0)(2^1 \lambda^1)|\lambda|^{-2} \\
+ B(1,0,1,1)(2^1 \lambda^1)|\lambda|^{-2} \alpha \Delta) \\
\times \sum_{\varepsilon,\delta} B(\sigma, \varepsilon, \delta, \nu) \prod_{0 \leq p, q \leq \sigma} (2^p \partial_p)^{\varepsilon_p} (2^q \lambda^q)^{\delta_q} |\lambda|^{-2|\delta|} \alpha^\nu \Delta^\nu,
\]

from which we note that, since the operators \( \partial_p \) commute with both \( \tau \) and \( \alpha \), it is sufficient to prove that

\[
\alpha \Delta^p \alpha^\nu \Delta^\nu = \sum_{t=0}^{\nu+1} S(\nu, p, t) \alpha^t \Delta^t, \tag{B.11}
\]

where the \( S(\nu, p, t) \) are polynomials in \( \tau \) and \( \tau^{-1} \). Now,

\[
\alpha \Delta^p \alpha^\nu \Delta^\nu = \alpha \tau^p \Delta^\nu = (\tau^p \alpha - p \tau^p) \Delta^\nu = \tau^p \alpha \Delta^\nu - p \tau^p \Delta^\nu,
\]

and the second summand is already of the form (B.11), since \( p \tau^p \Delta = p \tau^p (1 - \tau^{-1}) \).

For the former, we need to show that \( \alpha \Delta^\nu \Delta^\nu \) is also of the form (B.11). However, by (B.7) (with \( \mu = 0 \)),

\[
\alpha \Delta^\nu \Delta^\nu = \alpha \sum_{r=0}^{\nu} Q(\nu, r) \alpha^r \Delta^{r+1} = \sum_{r=0}^{\nu} (Q(\nu, r) \alpha + \tilde{Q}(\nu, r)) \alpha^r \Delta^{r+1} \\
= \sum_{r=0}^{\nu} Q(\nu, r) \alpha^{r+1} \Delta^{r+1} + \sum_{r=0}^{\nu} \tilde{Q}(\nu, r) \alpha^r \Delta^{r+1},
\]

and the second summand is already of the form (B.11), since \( p \tau^p \Delta = p \tau^p (1 - \tau^{-1}) \).

For the former, we need to show that \( \alpha \Delta^\nu \Delta^\nu \) is also of the form (B.11). However, by (B.7) (with \( \mu = 0 \)),

\[
\alpha \Delta^\nu \Delta^\nu = \alpha \sum_{r=0}^{\nu} Q(\nu, r) \alpha^r \Delta^{r+1} = \sum_{r=0}^{\nu} (Q(\nu, r) \alpha + \tilde{Q}(\nu, r)) \alpha^r \Delta^{r+1} \\
= \sum_{r=0}^{\nu} Q(\nu, r) \alpha^{r+1} \Delta^{r+1} + \sum_{r=0}^{\nu} \tilde{Q}(\nu, r) \alpha^r \Delta^{r+1},
\]

and the second summand is already of the form (B.11), since \( p \tau^p \Delta = p \tau^p (1 - \tau^{-1}) \).
and the first sum is of the form (B.11). For the latter, we prove
\[ \alpha^r \Delta^{r+1} = \sum_{s=0}^{r} R(r, s) \alpha^s \Delta^s, \tag{B.12} \]
for \( R(r, s) \) polynomials in \( \tau \) and \( \tau^{-1} \). Clearly, (B.12) is true for \( r = 0 \), as \( \Delta = 1 - \tau^{-1} \), so we assume it is true for \( 0, \ldots, r \). Hence
\[
\begin{align*}
\alpha^{r+1} \Delta^{r+2} &= \alpha \left( \sum_{s=0}^{r} R(r, s) \alpha^s \Delta^s \right) \Delta = \sum_{s=0}^{r} \left( R(r, s) \alpha + \tilde{R}(r, s) \right) \alpha^s \Delta^{s+1} \\
&= \sum_{s=0}^{r} R(r, s) \alpha^{s+1} \Delta^{s+1} + \sum_{s=0}^{r} \tilde{R}(r, s) \alpha^s \Delta^{s+1} \\
&= \sum_{s=0}^{r} R(r, s) \alpha^{s+1} \Delta^{s+1} + \sum_{s=0}^{r} \tilde{R}(r, s) \sum_{t=0}^{s} R(s, t) \alpha^t \Delta^t,
\end{align*}
\]
which is of the desired form. We have used (B.12) in the last equality.

We can therefore conclude (B.10), which for \( \sigma = d_k \) reads
\[
\prod_{p=1}^{d_k} (1 + 2^j \partial_{w_k^p}) = \sum_{0 \leq \nu \leq |\delta|} B(d_k, \varepsilon, \delta, \nu) \prod_{0 \leq p, q \leq d_k} (2^j \partial_{(k,p)})^{\varepsilon_p} (2^j \lambda_k^q)^{\delta_q} |\lambda_k|^{-2|\delta|} \alpha_k^\nu \Delta_k^\nu. \tag{B.13}
\]

Combining (B.9) and (B.13) we obtain
\[
(1 + 2^{j+L_k} \partial_{|w_k|^2})^{n_k} \prod_{p=1}^{d_k} (1 + 2^j \partial_{w_k^p}) = \sum_{0 \leq \mu \leq n_k} \sum_{0 \leq \nu \leq \mu + |\delta|} A(n_k, d_k, \mu, \nu, \varepsilon, \delta) \frac{\alpha^{\mu+j} \Delta_k^{\mu+\nu+\delta}}{|\lambda_k|^{\mu+2|\delta|}} \prod_{1 \leq p, q \leq d_k} (2^j \partial_{(k,p)})^{\varepsilon_p} (2^j \lambda_k^q)^{\delta_q} \alpha_k^\nu \Delta_k^\nu, \tag{B.14}
\]
where the sum is taken over all \( d_k \)-multiindexes \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d_k), \delta = (\delta_1, \ldots, \delta_d_k) \) with
\[ 0 \leq \varepsilon_p, \delta_q \leq 1, \ p, q = 1, \ldots, d_k; \quad \text{and} \quad |\delta| = \delta_1 + \ldots + \delta_d, \]
and
and the $\mathcal{A}(n_k, d_k, \mu, \nu, \varepsilon, \delta)$ are polynomials in $\tau_k$ and $\tau_k^{-1}$. Indeed, it is sufficient to observe that

$$\alpha^\nu \Delta^{\nu+\mu} \tau^p \alpha^\xi \Delta^\xi = \sum_{r=0}^{\nu+\xi} V(\nu, \mu, \xi, p, r) \alpha^r \Delta^{r+\mu},$$

as every scalar in (B.14) commute with the operators $\alpha$ and $\Delta$, and the $B(\sigma, \varepsilon, \delta, \nu)$ are linear combinations in $\tau^p$. To prove this, we first prove

$$\Delta^\mu \alpha^\xi \Delta^\xi = \sum_{t=0}^\xi U(\mu, \xi, t) \alpha^t \Delta^t. \tag{B.15}$$

We prove (B.15) by induction on $\mu$, being $\mu = 0$ trivial. Now, assuming (B.15) is true for $\mu$, and using the fact that $\Delta$ and $\tau$ commute, together with (B.7), we have

$$\Delta^{\mu+1} \alpha^\xi \Delta^\xi = \Delta \cdot \sum_{t=0}^\xi U(\mu, \xi, t) \alpha^t \Delta^t = \sum_{t=0}^\xi U(\mu, \xi, t) \Delta \alpha^t \Delta^t$$

$$= \sum_{t=0}^\xi U(\mu, \xi, t) \sum_{r=0}^t Q(t, r) \alpha^r \Delta^{r+\mu+1}$$

$$= \sum_{r=0}^\xi \left( \sum_{t=r}^\xi U(\mu, \xi, t) Q(t, r) \right) \alpha^r \Delta^{r+\mu+1},$$

which is in the form (B.15).

So we finally have, using (B.4) of Step 1 and (B.15),

$$\alpha^\nu \Delta^{\nu+\mu} \tau^p \alpha^\xi \Delta^\xi = \alpha^\nu \tau^p \Delta^{\nu+\mu} \alpha^\xi \Delta^\xi = \left( \sum_{r=0}^{\nu} D(\nu, p, r) \alpha^r \Delta^{r+\mu} \right) \alpha^\xi \Delta^\xi$$

$$= \sum_{r=0}^{\nu} D(\nu, p, r) \alpha^r \Delta^{r+\mu} \alpha^\xi \Delta^\xi = \sum_{r=0}^{\nu} \sum_{s=0}^\xi D(\nu, p, r) \alpha^r U(r, \xi, s) \alpha^s \Delta^{s+r+\mu}$$

$$= \sum_{r=0}^{\nu} \sum_{s=0}^\xi \sum_{t=0}^{r+s} D(\nu, p, r) \alpha^t \Delta^{t+\mu},$$

where we have used the observation (B.6) after Step 2. The last sum is already in the desired form.
Bibliography


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