EQUIVARIANT LOOPS OF HAMILTONIAN DIFFEOMORPHISMS

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ABSTRACT. We use Weinstein's homomorphism to detect non trivial loops of Hamiltonian diffeomorphism in the total space of a Hamiltonian fibration. In particular we detect a non-contractible loop in the group of Hamiltonian diffeomorphisms of a ruled surface.

1. Introduction

Consider (M,ω) a closed symplectic manifold and G a compact connected Lie group. Assume that G acts on M via Hamiltonian diffeomorphisms. Let $P \longrightarrow B$ be a principal G-bundle, where the base space is a closed symplectic manifold (B,β). By results of W. Thurston [13] and A. Weinstein [15] the associated bundle $E := P \times_G M \longrightarrow B$ becomes a Hamiltonian fibration with fiber (M,ω). That is, the structure group of the fibration is the group $\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms of (M,ω). Such a fibration satisfies that E carries a symplectic form Ω, and the inclusion of the fiber $\iota : M \longrightarrow E$ is a symplectic embedding.

We are interested in the case when a nontrivial loop of Hamiltonian diffeomorphisms of the fiber (M,ω) induces a nontrivial loop of Hamiltonian diffeomorphisms of the total space (E,Ω) of the fibration. Of course we must restrict to G-equivariant Hamiltonian diffeomorphisms of (M,ω). Denote by $\mathrm{Ham}_G(M,\omega)$ the subgroup of $\mathrm{Ham}(M,\omega)$ that consists of G-equivariant Hamiltonian diffeomorphisms. If $\mathrm{Ham}_G(M,\omega)_0$ stands for the connected component of $\mathrm{Ham}_G(M,\omega)$ that contains the identity map, we show in Proposition 3.10 that every G-equivariant Hamiltonian diffeomorphism of (M,ω) induces a Hamiltonian diffeomorphism of (E,Ω). Here the topology of the group of Hamiltonian diffeomorphisms is the C^∞-topology. Thus there is a group homomorphism

$$\Phi : \mathrm{Ham}_G(M,\omega)_0 \longrightarrow \mathrm{Ham}(E,\Omega).$$

Now a useful tool to study the fundamental group of $\mathrm{Ham}(M,\omega)$ is Weinstein's homomorphism,

$$\mathcal{A} : \pi_1(\mathrm{Ham}(M,\omega)) \longrightarrow \mathbb{R}/\Gamma(M,\omega)$$

defined by A. Weinstein in [14]. Here $\Gamma(M,\omega)$ is a subgroup of $\mathbb{R}$ and is called the group of spherical periods of the symplectic manifold (M,ω). In Section 2 we will review the definitions of the homomorphism $\mathcal{A}$ and the group $\Gamma(M,\omega)$.

Key words and phrases. Hamiltonian diffeomorphism, Hamiltonian fibrations, Weinstein's homomorphism.

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In the case of the Hamiltonian fibration $\pi : E \longrightarrow B$ with fiber (M,ω), we will see in Section 3 that $\Gamma(M,\omega)$ is a subgroup of $\Gamma(E,\Omega)$ which in turn defines a group homomorphism

$$\lambda : \mathbb{R}/\Gamma(M,\omega) \longrightarrow \mathbb{R}/\Gamma(E,\Omega).$$

as $\lambda(x+\Gamma(M,\omega)) = x+\Gamma(E,\Omega)$.

We make a comment on the notation. Throughout this article we will use ψ to denote a single diffeomorphism and also to denote a loop of diffeomorphisms $\psi = \{\psi_t\}_{0\leq t\leq 1}$ based at the identity. In the latter case, ψ will also denote the homotopy class in the fundamental group.

Theorem 1.1. *Let (M,ω) be a closed symplectic manifold and G a compact connected Lie group that acts on M via Hamiltonian diffeomorphisms. If ψ is a class in $\pi_1(\mathrm{Ham}(M,\omega))$, which is represented by a G-equivariant loop and $\Phi_*(\psi)$ is the induced loop in $\mathrm{Ham}(E,\Omega)$, then*

$$\lambda(\mathcal{A}(\psi)) = \mathcal{A}(\Phi_*(\psi))$$

in $\mathbb{R}/\Gamma(E,\Omega)$.

Is important to make clear that in Theorem 1.1, ψ is a loop of G-equivariant Hamiltonian diffeomorphisms and we think about it as an element of $\pi_1(\mathrm{Ham}(M,\omega))$ and *not* as an element of $\pi_1(\mathrm{Ham}_G(M,\omega))$.

Thus in order to know if $\Phi_*(\psi)$ represents a nonzero element in $\pi_1(\mathrm{Ham}(E,\Omega))$ we need to know $\mathcal{A}(\psi)$ and the kernel of λ. The kernel of λ is just $\Gamma(E,\Omega)/\Gamma(M,\omega)$. And for some special cases of homotopy classes ψ, $\mathcal{A}(\psi)$ can be be calculated easily. For instance $\mathcal{A}(\psi)$ can be calculated easily when ψ is a Hamiltonian circle action. See [15] for the details.

Consider $\pi : E \longrightarrow B$ a Hamiltonian fibration with fiber M. If $\pi_2(B)$ is trivial, it follows from the long exact sequence of homotopy groups of the fibration, that the induced map $\iota_* : \pi_2(M) \longrightarrow \pi_2(E)$ is surjective. In this case we have $\Gamma(M,\omega) = \Gamma(E,\Omega)$, and $\lambda : \mathbb{R}/\Gamma(M,\omega) \longrightarrow \mathbb{R}/\Gamma(E,\Omega)$ is the identity map.

Corollary 1.2. *Let $\pi : E \longrightarrow B$ be a Hamiltonian fibration with fiber M as in Theorem 1.1. If $\pi_2(B) = 0$, then*

$$\mathcal{A}(\psi) = \mathcal{A}(\Phi_*(\psi))$$

in $\mathbb{R}/\Gamma(E,\Omega)$.

There is another case in which the map λ is interesting. IsIf $\mathrm{Ham}_G(M,\omega)_0$ stands for the connected component of $\mathrm{Ham}_G(M,\omega)$ the case when (M,ω) is weakly exact, this means that $\Gamma(M,\omega) = 0$. In this case $\lambda : \mathbb{R} \longrightarrow \mathbb{R}/\Gamma(E,\Omega)$ is the projection map.

Example 3. Consider $B = \Sigma_g$ a Riemann surface with a fixed symplectic form β and $g \geq 1$. Thus the frame bundle $P \longrightarrow B = \Sigma_g$ is a S^1-principal bundle. Consider $\mathbb{C}P^1$ with the canonical S^1-action which is a Hamiltonian action. Hence we get the associated fibration, $\pi : E \longrightarrow \Sigma_g$, a $\mathbb{C}P^1$-fibration. Thus the total space, E, is a ruled surface.

Assume that $(\mathbb{C}P^1,\omega)$ has area 1. Let ψ be the same circle action on $\mathbb{C}P^1$ as above. Hence it is a Hamiltonian circle action. Therefore $\psi \in \pi_1(\mathrm{Ham}(\mathbb{C}P^1,\omega))$ and ψ is an equivariant loop. Then by [15], we have that $\Gamma(\mathbb{C}P^1,\omega)$ equals $\mathbb{Z}$ and $\mathcal{A}(\psi) = 1/2$. Moreover it is well known that ψ has order 2 in $\pi_1(\mathrm{Ham}(\mathbb{C}P^1,\omega))$.

Therefore from Corollary 1.2 we have that $\Phi_*(\psi)$ has even order in $\pi_1(\mathrm{Ham}(E,\Omega))$.

Remark. One might think that the symplectic form of the total space (E, Ω) is too restrictive. However Weinstein's morphism behaves well under some changes on the symplectic form. For consider ω_0 and ω_1 two symplectic forms on M, and $\psi : M \longrightarrow M$ a diffeomorphism such that $\psi^*(\omega_0) = \omega_1$. Then, under conjugation by ψ, $\mathcal{A}$ detects the same elements of $\pi_1(\mathrm{Ham}(M, \omega_0))$ and $\pi_1(\mathrm{Ham}(M, \omega_1))$.

In particular the result in the above example, for the ruled surface (E, Ω), holds for any symplectic form on E. This follows from the work of T. J. Li and A. Liu [7].

Is important to distinguish the case when G is the trivial group. In this case the total space of the fibration becomes $E = B \times M$. In this special case we will consider (E, Ω) to be the space $(M \times N, \omega \oplus \eta)$ where (N, η) is another closed symplectic manifold and $\omega \oplus \eta$ stands for the symplectic form $\pi_M^*(\omega) + \pi_N^*(\eta)$, where π_M and π_N are the projection maps. Observe that $\mathrm{Ham}_G(M, \omega) = \mathrm{Ham}(M, \omega)$ and $\mathrm{Ham}(E, \Omega) = \mathrm{Ham}(M \times N, \omega \oplus \eta)$.

In this special case it is well known that if ψ is a Hamiltonian diffeomorphism of (M, ω) then $\psi \times \mathrm{id}_N$ is a Hamiltonian diffeomorphism of $(M \times N, \omega \oplus \eta)$. In this case the group homomorphism Φ is replaced by the group homomorphism

$$\Psi : \mathrm{Ham}(M, \omega) \longrightarrow \mathrm{Ham}(M \times N, \omega \oplus \eta)$$

defined by $\Psi(\psi) = \psi \times \mathrm{id}_N$. We will see in Lemma 3.12 that $\Gamma(M, \omega) \subset \Gamma(M \times N, \omega \oplus \eta)$. Hence there is a well defined map

$$\kappa : \mathbb{R}/\Gamma(M, \omega) \longrightarrow \mathbb{R}/\Gamma(M \times N, \omega \oplus \eta)$$

defined as $\kappa(x + \Gamma(M, \omega)) = x + \Gamma(M \times M, \omega \oplus \omega)$ that plays the role of the map λ of Theorem 1.1.

Theorem 1.4. *Let (M, ω) and (N, η) be closed symplectic manifolds, then $\mathcal{A} \circ \tau_* = \kappa \circ \mathcal{A}$. That is, the following diagram commutes*

$$\begin{array}{ccc} \pi_1(\mathrm{Ham}(M, \omega)) & \xrightarrow{\ \Psi_*\ } & \pi_1(\mathrm{Ham}(M \times N, \omega \oplus \eta)) \\ \mathcal{A} \downarrow & & \downarrow \mathcal{A} \\ \mathbb{R}/\Gamma(M, \omega) & \xrightarrow{\ \kappa\ } & \mathbb{R}/\Gamma(M \times N, \omega \oplus \eta) \end{array}$$

Note that Theorem 1.4 is the analog of that of A. Pedroza [10], where Seidel's representation is used instead of Weinstein's homomorphism. Moreover Theorem 1.4 is more general than that of [10] since it does not require (M, ω) to be monotone and $\pi_2(N) = 0$. See also R. Leclercq [6], where he uses Seidel's representation and removes the hypothesis $\pi_2(N) = 0$.

There are several cases when the map κ of Theorem 1.4 is the identity map. For instance consider the case when (N, η) is weakly exact, that is

$$\int_{S^2} f^* \eta = 0$$

for every smooth function $f : S^2 \longrightarrow N$. In this case the group of spherical periods of (N, η) is trivial, $\Gamma(N, \eta) = 0$. Then by Lemma 3.12, $\Gamma(M \times N, \omega \oplus \eta) = \Gamma(M, \omega)$. Therefore, $\mathcal{A}(\psi) = \mathcal{A}(\psi \times \mathrm{id}_N)$ for every loop ψ in $\mathrm{Ham}(M, \omega)$.

Corollary 1.5. *Let* (M,ω) *and* (N,η) *be a closed symplectic manifold such that* (N,η) *is weakly exact. Then*

$$\mathcal{A}(\psi) = \mathcal{A}(\psi \times \mathrm{id}_N)$$

for every ψ *in* $\pi_1(\mathrm{Ham}(M,\omega))$.

In the case when $(M,\omega) = (N,\eta)$, we have a strong relationship between the spherical period groups. We will see in Lemma 3.12 that $\Gamma(M \times M, \omega \oplus \omega) = \Gamma(M,\omega)$. Thus it follows that the group homomorphism κ defined above is the identity map.

Corollary 1.6. *Let* (M,ω) *be a closed symplectic manifold. If* ψ *is a loop in* $\mathrm{Ham}(M,\omega)$*, then*

$$\mathcal{A}(\psi) = \mathcal{A}(\psi \times \mathrm{id}_M)$$

in $\mathbb{R}/\Gamma(M,\omega)$. *In particular if* $\mathcal{A} : \pi_1(\mathrm{Ham}(M,\omega)) \longrightarrow \mathbb{R}/\Gamma(M,\omega)$ *is injective, then the map* $\Psi_* : \pi_1(\mathrm{Ham}(M,\omega)) \longrightarrow \pi_1(\mathrm{Ham}(M \times M, \omega \oplus \omega))$ *is injective.*

There are only a few special cases of symplectic manifolds (M,ω) for which the group $\pi_1(\mathrm{Ham}(M,\omega))$ is completely known. In the case of the symplectic manifolds $(S^2,\omega), (S^2 \times S^2, \omega \oplus \omega)$ and $(\mathbb{C}P^2, \omega_{FS})$, the group $\pi_1(\mathrm{Ham}(M,\omega))$ is known, [3]; and in all three cases Weinstein's homomorphism is injective. The group $\pi_1(\mathrm{Ham}(M,\omega))$ is also known in the cases $(S^2 \times S^2, \lambda\omega \oplus \omega)$, for $1 < \lambda \leq 2$, and the one point blow up of $(S^2 \times S^2, \omega \oplus \omega)$. See [1], [5] and [9] for details.

Finally, using the fact that $\mathcal{A}$ is a group homomorphism we also obtain information about the induced loop $\psi \times \psi$ in $\mathrm{Ham}(M \times M, \omega \oplus \omega)$.

Corollary 1.7. *Let* $\psi \in \pi_1(\mathrm{Ham}(M,\omega))$*, then*

$$\mathcal{A}(\psi \times \psi) = 2\mathcal{A}(\psi)$$

in $\mathbb{R}/\Gamma(M,\omega)$.

2. The group of Hamiltonian diffeomorphisms and Weinstein's homomorphism

We recall the definition of the group, $\mathrm{Ham}(M,\omega)$, of Hamiltonian diffeomorphisms of a closed symplectic manifold (M,ω). First, the group of symplectic diffeomorphisms $\mathrm{Symp}(M,\omega)$ of (M,ω) is defined as the set of all diffeomorphisms $\psi : M \longrightarrow M$ such that $\psi^*(\omega) = \omega$. The group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M,\omega)$ forms a subgroup of $\mathrm{Symp}(M,\omega)$. In order to define the group $\mathrm{Ham}(M,\omega)$ we recall the following fact about isotopies. Let $\{\psi_t\}_{0\leq t\leq 1}$ be an isotopy, that is a path in $\mathrm{Diff}(M)$. Then the isotopy determines a time-dependent vector field X_t on M defined as

$$X_t(p) := \left.\frac{d}{ds}\right|_{s=t} \psi_s \circ \psi_t^{-1}(p). \tag{1}$$

A symplectic diffeomorphism ψ is called **Hamiltonian diffeomorphism** if there exists an isotopy $\{\psi_t\}_{0\leq t\leq 1}$ of symplectic diffeomorphisms that starts at the identity map, id_M, and ends at ψ; and a smooth map $H : S^1 \times M \longrightarrow \mathbb{R}$ such that

$$\iota(X_t)\omega = -dH_t.$$

Here X_t is the time-dependent vector field induced by the isotopy of symplectic diffeomorphisms $\{\psi_t\}$ as in (1). Further facts about the groups $\mathrm{Symp}(M,\omega)$ and $\mathrm{Ham}(M,\omega)$ can be found in L. Polterovich [11].

Consider the group homomorphism $I_\omega : \pi_2(M) \longrightarrow \mathbb{R}$ defined by

$$I_\omega(\alpha) := \int_\alpha \omega = \int_{S^2} \alpha^*(\omega).$$

Here we used the fact that any element of $\pi_2(M)$ can be realized by a smooth map $\alpha : S^2 \longrightarrow M$. The image of the homomorphism I_ω is called the **group of spherical periods** of (M,ω) and is denoted by $\Gamma(M,\omega) \subset \mathbb{R}$.

Let $\mathcal{L}_0(M)$ be the space of contractible loops in M. Fix a smooth normalized function $H : S^1 \times M \longrightarrow \mathbb{R}$, that is

$$\int_M H_t \; \omega^n = 0$$

for each t and the dimension of M is assumed to be $2n$. Then the action functional $\mathcal{A}_H : \mathcal{L}_0(M) \longrightarrow \mathbb{R}/\Gamma(M,\omega)$ is defined by

$$\mathcal{A}_H(\gamma) := -\int_{D_\gamma} \omega + \int_{S^1} H_t(\gamma(t))dt,$$

where D_γ is the image of any smooth map $D^2 \longrightarrow M$ with boundary the contractible loop γ.

Assume that the path of Hamiltonian diffeomorphisms, $\psi = \{\psi_t\}$, induced by H is a loop based at the identity map. That is $\psi_0 = \psi_1 = \mathrm{id}_M$. Note that in this case $\{\psi_t(p)\}$ is a loop in M for any $p \in M$. Furthermore, the loop $\{\psi_t(p)\}$ is contractible in M.

Theorem 2.8 (Lalonde, McDuff, Polterovich, [4]). *Let (M,ω) be a closed symplectic manifold and $\{\psi_t\}$ a loop of Hamiltonian diffeomorphisms based at the identity map. Then for any point $p \in M$, the loop $\{\psi_t(p)\}$ is contractible in M.*

Is known that for such Hamiltonian function H, the loops $\{\psi_t(p)\}$ are critical points of the action $\mathcal{A}_H$. Moreover these critical points minimized the functional $\mathcal{A}_H$, thus $\mathcal{A}_H(\{\psi_t(p)\}) = \mathcal{A}_H(\{\psi_t(q)\})$ for any points $p,q \in M$. In [14], A. Weinstein showed that the functional $\mathcal{A}_H$ depends only on the homotopy type of the loop $\psi = \{\psi_t\}$ of Hamiltonian diffeomorphisms in $\mathrm{Ham}(M,\omega)$ induced by H. Hence Weinstein's homomorphism

$$\mathcal{A} : \pi_1(\mathrm{Ham}(M,\omega)) \longrightarrow \mathbb{R}/\Gamma(M,\omega)$$

is defined as $\mathcal{A}(\psi) = \mathcal{A}_H(\{\psi_t(p)\})$, for ψ in $\pi_1(\mathrm{Ham}(M,\omega))$.

3. Hamiltonian fibrations and induced Hamiltonian diffeomorphisms

A **symplectic fibration** $\pi : E \longrightarrow B$ with fiber (M,ω) is a fiber bundle whose structure group is the group of symplectomorphism, $\mathrm{Symp}(M,\omega)$, of (M,ω). For $b \in B$ let $\iota_b : M \longrightarrow E$ be the inclusion of the fiber above b. Suppose that the base space of the fibration is a compact symplectic manifold (B,β). According to W. Thurston [13] (see also Theorem 6.3, p. 199 in [8]) if there is a cohomology class $a \in H^2(E)$ such that $\iota_b^*(a) = [\omega]$ for some $b \in B$, then there exists a closed 2-form $\tilde{\omega}$ on E that restricts to ω on every fiber. In addition the 2-form

$$\Omega = \tilde{\omega} + K\pi^*(\beta)$$

is a symplectic form on E for a positive large K. In this case $\iota_b : (M, \omega) \longrightarrow (E, \Omega)$ is a symplectic embedding of the fiber. That is $\iota_b(M)$ is a symplectic submanifold of the total space E. Although different values of K determine different symplectic forms, the symplectic structure on E does not change.

As in the introduction let G be a compact connected Lie group and $P \longrightarrow B$ a principal G-bundle such that the base space is a symplectic manifold (B, β). Assume that G acts on (M, ω) via a Hamiltonian action. Let $E := P \times_G M$. Then the associated fibration

$$\pi : E \longrightarrow B$$

is not only a symplectic fibration, it is also a **Hamiltonian fibration** with fiber M. That is, the structure group of the fibration is $\text{Ham}(M, \omega)$. According to A. Weinstein [15] (see also Theorem 6.17, p. 209 in [8]) in a Hamiltonian fibration there exists a closed 2-form $\tilde{\omega}$ on E that restricts to ω_b on every fiber of π. Therefore $\Omega := \tilde{\omega} + K\pi^*(\beta)$ is a symplectic form on the total space of the fibration E for a positive large value of K.

Define the group

$$\text{Ham}_G(M, \omega) = \{\psi \in \text{Ham}(M, \omega) \mid \psi g = g\psi \text{ for all } g \in G\}$$

of G-equivariant Hamiltonian diffeomorphisms of (M, ω), and $\text{Ham}_G(M, \omega)_0$ the connected component that contains the identity map. Denote by $\rho : P \times M \longrightarrow E = P \times_G M$ the projection map $\rho(x, p) = [x, p]$. Then each map $\psi \in \text{Ham}_G(M, \omega)$ defines a map on the total space of the fibration $\tilde{\psi} : E \longrightarrow E$ via

$$\begin{array}{ccc} P \times M & \xrightarrow{\text{id}_P \times \psi} & P \times M \\ \downarrow \rho & & \downarrow \rho \\ E & \xrightarrow[\tilde{\psi}]{} & E \end{array}$$

That is $\rho \circ (\text{id}_P \times \psi) = \tilde{\psi} \circ \rho$. Moreover note that $\tilde{\psi}$ is a bundle map of the fibration $\pi : E \longrightarrow B$, this means that $\pi \circ \tilde{\psi} = \pi$.

Lemma 3.9. *Let (M, ω) be a closed symplectic manifold and G a compact connected Lie group that acts on M via Hamiltonian diffeomorphisms. If $\{\psi_t\}$ is an isotopy of G-equivariant symplectic diffeomorphisms of (M, ω), then $\{\tilde{\psi}_t\}$ is an isotopy of symplectic diffeomorphisms of (E, Ω)*

Proof. Let $\{\psi_t\}_{0 \leq t \leq 1}$ be a path of symplectic diffeomorphisms that are G-equivariant. Since the maps $\psi_t : M \longrightarrow M$ are G-equivariant, they descend to maps $\tilde{\psi}_t : E \longrightarrow E$. Recall that $\tilde{\psi}_t$ are bundle maps, that is $\pi \circ \tilde{\psi}_t = \pi$. Then

$$\tilde{\psi}_t^*(\pi^*(\beta)) = \pi^*(\beta). \tag{2}$$

According the the proof of Weinstein's Theorem (p. 209 in [8]), the differential form $\rho^*(\tilde{\omega})$ splits in $\Omega^2(P \times M)$. Moreover ω is the only term that comes from

$\Omega^*(M)$. Thus let $\alpha \in \Omega^2(P)$ such that $\rho^*(\tilde{\omega}) = \alpha + \omega$. Then

$$\begin{aligned}
\rho^* \circ \tilde{\psi}_t^*(\tilde{\omega}) &= (\mathrm{id}_P \times \psi_t)^* \circ \rho^*(\tilde{\omega}) \\
&= (\mathrm{id}_P \times \psi_t)^*(\alpha + \omega) \\
&= \alpha + \psi_t^*(\omega) \\
&= \alpha + \omega \\
&= \rho^*(\tilde{\omega}).
\end{aligned}$$

The map ρ is the projection map of a principal G-bundle, and the group is compact connected. Thus the induced map on differential forms $\rho^* : \Omega^*(E) \longrightarrow \Omega^*(P \times M)$ is injective. Therefore

$$\tilde{\psi}_t^*(\tilde{\omega}) = \tilde{\omega}. \tag{3}$$

Hence from (2) and (3) we see that

$$\tilde{\psi}_t^*(\Omega) = \tilde{\psi}_t^*(\tilde{\omega} + K\pi^*(\beta)) = \Omega$$

for all t. That is $\{\tilde{\psi}_t\}_{0 \le t \le 1}$ is an isotopy in $\mathrm{Symp}(E, \Omega)$. □

Recall that any G-invariant function $F : M \longrightarrow \mathbb{R}$ induces a function $\tilde{F} : E \longrightarrow \mathbb{R}$. So let $F' : P \times M \longrightarrow \mathbb{R}$ be defined by $F' = F \circ \pi_M$ where $\pi_M : P \times M \longrightarrow M$ is the projection map. Then the function F' is also G-invariant, thus it defines a function $\tilde{F} : E \longrightarrow \mathbb{R}$ by $F' = \tilde{F} \circ \rho$.

Now we prove that G-equivariant Hamiltonian diffeomorphisms of (M, ω) descend to Hamiltonian diffeomorphisms of (E, Ω). Recall that $\mathrm{Ham}_G(M, \omega)_0$ stands for the connected component of $\mathrm{Ham}_G(M, \omega)$

Proposition 3.10. *Let (M, ω), (E, Ω) and G be as above. If $\psi \in \mathrm{Ham}_G(M, \omega)_0$, then the map $\tilde{\psi}$ belongs to $\mathrm{Ham}(E, \Omega)$. Moreover if H_t is the Hamiltonian function of ψ, then the Hamiltonian function of $\tilde{\psi}$ is $\tilde{H}_t$.*

Proof. Let $\{\psi_t\}$ be a path in $\mathrm{Ham}_G(M, \omega)_0$ such that $\psi_0 = \mathrm{id}_M$ and $\psi_1 = \psi$. By Lemma 3.9, we have that $\{\tilde{\psi}_t\}$ is an isotopy of symplectic diffeomorphisms of (E, Ω) that starts at the identity and ends at $\tilde{\psi}$.

Let X_t and $H_t : M \longrightarrow \mathbb{R}$ be the time-dependent vector field and Hamiltonian function associated with $\{\psi_t\}$. Since the isotopy $\{\psi_t\}$ consists of G-equivariant maps if follows that the time-dependent vector field X_t and Hamiltonian function H_t are G-equivariant. Consider $X'_t = 0 \oplus X_t$ as a time-dependent vector field on $P \times M$. Since $\rho : P \times M \longrightarrow E$ is a principal G-bundle and X'_t is G-invariant in $P \times M$ there is a time-dependent vector field $\tilde{X}_t$ on E which is ρ-related with X'_t, that is $\rho_*(X'_t) = \tilde{X}_t$. We will see that $\tilde{X}_t$ is the time-dependent vector field related to the isotopy $\{\tilde{\psi}_t\}$.

Recall that for $(x, p) \in P \times M$,

$$X'_t(x, p) = \left.\frac{d}{ds}\right|_{s=0} (\mathrm{id}_P \times \psi_{t+s}) \circ (\mathrm{id}_P \times \psi_t)^{-1}(x, p).$$

Therefore

$$\begin{aligned}
\tilde{X}_t(\rho(x,p)) = \rho_*(X'_t(x,p)) &= \rho_*\left(\left.\frac{d}{ds}\right|_{s=0} (\mathrm{id}_P \times \psi_{t+s}) \circ (\mathrm{id}_P \times \psi_t)^{-1}(x,p) \right) \\
&= \left.\frac{d}{ds}\right|_{s=0} \rho \circ (\mathrm{id}_P \times \psi_{t+s}) \circ (\mathrm{id}_P \times \psi_t)^{-1}(x,p) \\
&= \left.\frac{d}{ds}\right|_{s=0} \tilde{\psi}_{t+s} \circ \rho \circ (\mathrm{id}_P \times \psi_t^{-1})(x,p) \\
&= \left.\frac{d}{ds}\right|_{s=0} \tilde{\psi}_{t+s} \circ \tilde{\psi}_t(\rho(x,p)).
\end{aligned}$$

That is, the symplectic isotopy $\{\tilde{\psi}_t\}$ induces the time-dependent vector field $\tilde{X}_t$ on E.

Since $H_t : M \longrightarrow \mathbb{R}$ is a G-invariant function, there is a function $\tilde{H}_t : E \longrightarrow \mathbb{R}$ such that $H'_t = \tilde{H}_t \circ \rho$, where H' is defined as $H'_t = H_t \circ \pi_M$. Therefore

$$\begin{aligned}
\iota(\tilde{X}_t)(\tilde{\omega} + K\pi^*(\beta)) &= \iota(\rho_*(X'_t))(\tilde{\omega}) + K\iota(\rho_*(X'_t))(\pi^*(\beta)) \\
&= \iota(\rho_*(0 \oplus X_t))(\tilde{\omega}) + K\iota(\rho_*(X'_t))(\pi^*(\beta)) \\
&= -d\tilde{H}_t + 0.
\end{aligned}$$

That is, $\{\tilde{\psi}_t\}$ is a Hamiltonian isotopy with moment map $\tilde{H}_t$. Therefore $\tilde{\psi} = \tilde{\psi}_1$ is a Hamiltonian diffeomorphism. □

It follows from the above result that

$$\Phi : \mathrm{Ham}_G(M,\omega)_0 \longrightarrow \mathrm{Ham}(E,\Omega)$$

defined by $\Phi(\psi) = \tilde{\psi}$ is a well defined map. Moreover, the map Φ is a group homomorphism.

In the Cartesian product case, it is well known that if ψ is a Hamiltonian diffeomorphism of (M,ω), then $\psi \times \mathrm{id}_N$ is also a Hamiltonian diffeomorphism of $(M \times N, \omega \oplus \eta)$. Moreover it is well known what is the relationship with the corresponding moment maps. That is, if ψ is a loop of Hamiltonian diffeomorphisms of (M,ω) with Hamiltonian function $H_t : M \longrightarrow \mathbb{R}$, then $\psi \times \mathrm{id}_N$ is a loop of Hamiltonian diffeomorphisms of $(M \times N, \omega \oplus \eta)$ with Hamiltonian function $H'_t : M \times N \longrightarrow \mathbb{R}$ defined by

$$H'_t(p,q) = H_t(p).$$

Now we define the maps λ and κ of the introduction. In order to do that, we need relationship between the spherical period groups of (M,ω) and (E,Ω).

Lemma 3.11. *Let $\iota_* : \pi_2(M) \longrightarrow \pi_2(E)$ be the induced map of the fiber inclusion. Then for $\alpha \in \pi_2(M)$,*

$$I_\omega(\alpha) = I_\Omega(\iota_*(\alpha)).$$

Proof. Let $\alpha : S^2 \longrightarrow M$ in $\pi_2(M)$, such that α is smooth. Then $\iota_*(\alpha) = \iota \circ \alpha$. Then

$$(\iota \circ \alpha)^*(\Omega) = (\iota \circ \alpha)^*(\tilde{\omega} + K\pi^*(\beta)) = \alpha^* \circ \iota^*(\tilde{\omega} + K\pi^*(\beta)) = \alpha^*(\omega)$$

since $\tilde{\omega}$ restricts to ω on every fiber. Hence

$$I_\omega(\alpha) = \int_{S^2} \alpha^*(\omega) = \int_{S^2} (\iota \circ \alpha)^*(\Omega) = I_\Omega(\iota_*(\alpha)).$$

□

It follows from Lemma 3.11 that $\Gamma(M,\omega) \subset \Gamma(E,\Omega)$. Then there is a well defined group homomorphism

$$\lambda : \mathbb{R}/\Gamma(M,\omega) \longrightarrow \mathbb{R}/\Gamma(E,\Omega)$$

defined by $\lambda(x + \Gamma(M,\omega)) = x + \Gamma(E,\Omega)$.

Now in the case when G is the trivial group, that is, when we consider the Cartesian product we have the following relationship between the spherical period groups.

Lemma 3.12. *Let (M,ω) and (N,η) be closed symplectic manifolds. Then*

$$\Gamma(M \times N, \omega \oplus \eta) = \Gamma(M,\omega) + \Gamma(N,\eta).$$

Proof. Recall that $\pi_2(M \times N) = \pi_2(M) \oplus \pi_2(N)$. Thus any $\alpha \in \pi_2(M \times N)$ can be written as $\alpha = \alpha_1 \oplus \alpha_2$ where $\alpha_1 \in \pi_2(M)$ and $\alpha_2 \in \pi_2(N)$. Therefore

$$\int_\alpha \omega \oplus \eta = \int_{\alpha_1} \omega + \int_{\alpha_2} \eta.$$

Hence $\Gamma(M \times N, \omega \oplus \eta) = \Gamma(M,\omega) + \Gamma(N,\eta)$. □

Thus from Lemma 3.12, we have that $\Gamma(M,\omega) \subset \Gamma(M \times N, \omega \oplus \eta)$. Therefore the map

$$\kappa : \mathbb{R}/\Gamma(M,\omega) \longrightarrow \mathbb{R}/\Gamma(M \times N, \omega \oplus \eta)$$

defined by $\kappa(x + \Gamma(M,\omega)) = x + \Gamma(M \times N, \omega \oplus \eta)$ is a well-defined group homomorphism.

4. Proof of main results

First we recall the following fact. Let $\pi : E \longrightarrow B$ be a fiber bundle with fiber F, such that E, B and F are compact oriented. Then there is a class $e \in H^*(E)$ of degree equal to the dimension of F such that $\pi_*(e) = 1$. Here π_* stands for integration along the fiber. Moreover it follows that

$$\int_B \alpha = \int_E e \wedge \pi^*(\alpha)$$

for any $\alpha \in H^*(B)$.

Lemma 4.13. *Let $H : M \longrightarrow \mathbb{R}$ be a normalized G-equivariant function and $\tilde{H} : E \longrightarrow \mathbb{R}$ the induced function as in the previous section. Then $\tilde{H}$ is normalized.*

Proof. Let $e \in H^*(P)$ such that $\rho'_*(e) = 1$, where ρ' is the projection map of the principal G-bundle $P \longrightarrow B$. The commutative diagram of principal G-bundles

$$\begin{array}{ccc} P \times M & \xrightarrow{\pi_P} & P \\ \rho \downarrow & & \downarrow \rho' \\ E & \xrightarrow[\pi]{} & B \end{array}$$

induces the commutative diagram in cohomology

$$\begin{array}{ccc} H^*(P \times M) & \xleftarrow{\pi_P^*} & H^*(P) \\ \rho_* \downarrow & & \downarrow \rho'_* \\ H^*(E) & \xleftarrow[\pi^*]{} & H^*(B) \cdot \end{array}$$

Therefore $\rho_*(\pi_P^*(e)) = 1$. Then if we set $e_0 = \pi_P^*(e)$, we have $\rho_*(e_0) = 1$.

Let $2n$ be the dimension of M and $2m$ the dimension of B. Then by the fact announced at the beginning of this section with respect to the principal G-bundle $\rho : P \times M \longrightarrow E$, we have

$$\int_E \tilde{H}\Omega^{m+n} = \int_{P\times M} e_0 \wedge \rho^*(\tilde{H}\Omega^{m+n})$$

where $e \in H^*(P \times N)$ is as above. Recall that $\rho^*(\tilde{\omega})$ splits as $\alpha + \omega$. Since H is a normalized function we have that

$$\begin{aligned}
\int_{P\times M} e_0 \wedge \rho^*(\tilde{H}\Omega^{m+n}) &= \int_{P\times M} e_0 \wedge H'((\alpha+\omega) + K(\pi^*(\beta)+0))^{m+n} \\
&= \sum_{j=0}^{m+n} \binom{m+n}{j} \left(\int_P e \wedge (\alpha + K\pi^*(\beta))^j\right)\left(\int_M H\omega^{m+n-j}\right) \\
&= \binom{m+n}{m} \left(\int_P e \wedge (\alpha + K\pi^*(\beta))^m\right)\left(\int_M H\omega^n\right) \\
&= 0
\end{aligned}$$

Thus, $\tilde{H} : E \longrightarrow \mathbb{R}$ is a normalized function. □

Proof of Theorem 1.1. Let ψ be a loop in $\mathrm{Ham}_G(M,\omega)$ and H_t the corresponding normalized Hamiltonian function. By Proposition 3.10, $\tilde{\psi}$ is the induced loop on $\mathrm{Ham}(E,\Omega)$ with Hamiltonian function $H_t' : M \longrightarrow \mathbb{R}$. Recall that the Hamiltonian functions of ψ and $\tilde{\psi}$ are related by $H_t' = \tilde{H}_t \circ \rho$, where $\tilde{H}_t : P \times M \longrightarrow \mathbb{R}$ is defined by $\tilde{H}_t(p,q) = H_t(q)$. By the previous result we know that each $\tilde{H}_t$ is a normalized function.

Observe that for any $(p,q) \in P \times M$,

$$\begin{aligned}
\int_{S^1} H_t(\psi_t(q))dt &= \int_{S^1} H_t'(p,\psi_t(q))dt \\
&= \int_{S^1} \tilde{H}_t \circ \rho(p,\psi_t(q))dt \\
&= \int_{S^1} \tilde{H}_t(\tilde{\psi}_t(q))dt.
\end{aligned}$$

Fix $(p,q) \in P \times M$. Let D be the image of $u : D^2 \longrightarrow M$ such that the loop $\{\psi_t(q)\}$ is the boundary of D. Then the image of $u_0 : D^2 \longrightarrow P \times M$ defined by $u_0(z) = (p, u(z))$ is a disk bounded by the loop $\{p \times \psi_t(q)\}$. Let $\tilde{D} = \rho \circ u_0(D^2)$, then $\tilde{D}$ is bounded by the loop $\{\tilde{\psi}_t([p,q])\}$ in E. Therefore

$$\begin{aligned}
\int_D \omega &= \int_{\{p\}\times D} \alpha + \omega \\
&= \int_{\rho(D)} \tilde{\omega} \\
&= \int_{\tilde{D}} \tilde{\omega} + K\pi^*(\beta).
\end{aligned}$$

Hence, as real numbers we have that

$$-\int_D \omega + \int_{S^1} H_t(\psi_t(q))dt = -\int_{\tilde{D}} \tilde{\omega} + K\pi^*(\beta) + \int_{S^1} \tilde{H}_t(\tilde{\psi}_t(q))dt.$$

That is, $\lambda\mathcal{A}(\psi) = \mathcal{A}(\Phi(\psi))$ in $\mathbb{R}/\Gamma(E, \Omega)$. □

The proof of Theorem 1.4 is similar to the previous argument. We give it here just for the sake of completeness of the article.

Proof of Theorem 1.4. Let ψ be a Hamiltonian loop in (M, ω) with normalized Hamiltonian function H. Then $\psi \times \mathrm{id}_N$ is a Hamiltonian loop in $(M \times N, \omega \oplus \eta)$ with normalized moment map $H'(p, q) = H(p)$.

Let $(p, q) \in M \times N$. If D is the image of $u : D^2 \longrightarrow M$ such that the boundary is the curve $\{\psi_t(p)\}$, then the image D' of the map $u_0 : D^2 \longrightarrow M \times N$ defined by $u_0(z) = (u(z), q)$ has boundary $\{\psi_t(p) \times \{q\}\}$ in $M \times N$. Moreover

$$\int_D \omega = \int_{D'} \omega + \eta.$$

Note also that

$$\int_t H'(p, q)dt = \int_t H(p)dt.$$

Hence as real numbers we have

$$-\int_D \omega + \int_t H(p)dt = -\int_{D'} \omega + \eta + \int_t H'(p, q)dt.$$

Therefore in $\mathbb{R}/\Gamma(M \times N, \omega \oplus \eta)$ we have

$$\kappa(\mathcal{A}(\psi)) = \mathcal{A}(\psi \times \mathrm{id}_N) = \mathcal{A}(\tau(\psi)).$$

□

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