GENERALIZING THE LOCALIZATION FORMULA IN EQUIVARIANT COHOMOLOGY

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Abstract. We give a generalization of the Atiyah-Bott-Berline-Vergne localization theorem for the equivariant cohomology of a torus action. We replace the manifold having a torus action by an equivariant map of a compact connected Lie group. This provides a systematic method for calculating the Gysin homomorphism in ordinary cohomology of an equivariant map. As an example, we recover a formula of Akyildiz-Carrell for the Gysin homomorphism of flag manifolds.

Suppose $M$ is a compact oriented manifold on which a torus $T$ acts. The Atiyah-Bott-Berline-Vergne localization formula calculates the integral of an equivariant cohomology class on $M$ in terms of an integral over the fixed point set $M^T$. This formula has found many applications, for example, in analysis, topology, symplectic geometry, and algebraic geometry (see [2], [8], [10], [14]). Similar, but not entirely analogous, formulas exist in $K$-theory ([3]), cobordism theory ([12]), and algebraic geometry ([9]).

Taking cues from the work of Atiyah and Segal in $K$-theory [3], we state and prove a localization formula for a compact connected Lie group in terms of the fixed point set of a conjugacy class in the group. As an application, the formula can be used to calculate the Gysin homomorphism in ordinary cohomology of an equivariant map. For a compact connected Lie group $G$ with maximal torus $T$ and a closed subgroup $H$ containing $T$, we work out as an example the Gysin homomorphism of the canonical projection $f: G/T \to G/H$, a formula first obtained by Akyildiz and Carrell [1].

The application to the Gysin map in this article complements that of [14]. The previous article [14] shows how to use the ABBV localization formula to calculate the Gysin map of a fiber bundle. This article shows how to use the relative localization formula to calculate the Gysin map of an equivariant map.

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1. Borel-type localization formula for a conjugacy class

Suppose a compact connected Lie group $G$ acts on a manifold $M$. For $g \in G$, define $M^g$ to be the fixed point set of $g$:

$$M^g = \{ x \in M \mid g \cdot x = x \}.$$ 

The set $M^g$ is not $G$-invariant. The $G$-invariant subset it generates is

$$\bigcup_{h \in G} h \cdot (M^g) = \bigcup_{h \in G} M^{h^{-1}} = \bigcup_{k \in C(\mu)} M^k$$
where $C(g)$ is the conjugacy class of $g$. This suggests that for every conjugacy class $C$ in $G$, we consider the set $M^C$ of elements of $M$ that are fixed by at least one element of the conjugacy class $C$:

$$M^C = \cup_{g \in C} M^g.$$ 

Then $M^C$ is a closed $G$-subset of $M$ ([3], footnote 1, p. 532); however it is not always smooth. From now on we make the assumption that $M^C$ is smooth.

**Remark 1.1.** If $T$ is a maximal torus in the compact connected Lie group $G$ and $\dim T = \ell$, then

$$H^*(BG) = H^*(BT)^W_G = \mathbb{Q}[u_1, \ldots, u_{\ell}]^{W_G}.$$ 

Thus, $H^*(BG)$ is an integral domain. Let $Q$ be its field of fractions. For any $H^*(BG)$-module $W$, we define the localization of $W$ with respect to the zero ideal in $H^*(BG)$ to be

$$\tilde{W} := W \otimes_{H^*(BG)} Q.$$ 

It is easily verified that $W$ is $H^*(BG)$-torsion if and only if $\tilde{W} = 0$. For a $G$-manifold $M$, we call $H_T^*(M)$ the localized equivariant cohomology of $M$.

**Proposition 1.2.** Let $M$ be a $T$-manifold with a finite number of orbit types and no fixed points. Then $H_T^*(M)$ is torsion.

**Proof.** For $p$ in $M$ let $t_p$ be the Lie algebra of the stabilizer of $p$. Since the $T$ action on $M$ has finitely many orbit types and has no fixed points, it follows that only a finite number of proper subspaces $t_1, t_2, \ldots, t_r$ of $t$ can occur as $t_p$ for $p$ in $M$.

Let $v$ be a vector in $t$ not in the union of the $t_i$. Then the induced vector field $v_M$ in $M$ is nowhere zero. Consider the invariant 1-form $\eta$ on $M$ defined by $\eta(X) = \langle v, X \rangle$. Thus $\eta(v) = 1$ and $d_C(\eta) = d\eta - u_1$. Therefore

$$1 - \frac{d\eta}{u_1} = d_C \left( -\frac{\eta}{u_1} \right)$$

and

$$1 = \left\{ d_C \left( -\frac{\eta}{u_1} \right) \right\} \cdot \left( 1 - \frac{d\eta}{u_1} \right)^{-1}.$$ 

Note that

$$\left( 1 - \frac{d\eta}{u_1} \right)^{-1} = \frac{d\eta}{u_1} + \left( \frac{d\eta}{u_1} \right)^2 + \cdots + \left( \frac{d\eta}{u_1} \right)^k.$$ 

Since $d_C(d\eta) = -u_{ij} \partial_j \eta = u_{ij} d\eta \delta_j^i = 0$ it follows that

$$1 = d_C \left\{ -\frac{\eta}{u_1}, \left( 1 - \frac{d\eta}{u_1} \right)^{-1} \right\}.$$ 

Hence $H_T^*(M)$ is torsion. \qed

**Lemma 1.3.** If $H_T^*(M)$ is $H^*(BT)$-torsion, then $H_C^*(M)$ is $H^*(BG)$-torsion.

**Proof.** Recall that the map $\psi : H_C^*(M) \rightarrow H_T^*(M)$ is injective. Since $H_T^*(M)$ is $H^*(BT)$-torsion, there is $a \in H^*(BT)$ such that $a \cdot 1_{H_T^*(M)} = 0$. Consider the average of $a$ over the Weyl group $W$ of $G$ with respect to $T$,

$$\tilde{a} = \frac{1}{|W|} (a + \omega_1 a + \cdots + \omega_r a) \in H^*(BG).$$
Under $\psi$, the element $\tilde{a} \cdot 1_{H^*_G(M)}$ goes to
\[
\frac{1}{|W|}(\omega_1 a + \cdots + \omega_r a)1_{H^*_G(M)}.
\]
But $(\omega_j a)1_{H^*_G(M)} = \omega_j(a1_{H^*_G(M)}) = 0$ for any $j$. Thus $\tilde{a} \cdot 1_{H^*_G(M)} = 0$ in $H^*_G(M)$. \hfill $\square$

**Proposition 1.4.** Let $G$ be a compact connected Lie group acting on a compact manifold $M$. Let $U$ be a $G$-invariant set such that $U^C$ is empty, then the equivariant cohomology $H^*_G(U)$ is $H^*(BG)$-torsion.

**Proof.** The set $U^C$ can be described
\[
U^C = \{ p \in U : hSh^{-1} \subset G_p \text{ for some } h \in G \}
\]
where $S$ is the closure of the group generated by some $g \in C$.

Let $T$ a maximal torus containing $S$. For every $p \in M$, $T$ is not contained in $hG_p h^{-1}$ for all $h \in G$. Therefore the fixed point set of $M$ with respect to $T$ is empty.

Since $M$ is compact, $U$ has only a finite number of orbit types. Thus from Prop. 1.2 and Lemma 1.3 it follows that $H^*_G(U)$ is torsion. \hfill $\square$

In the rest of this section, “torsion” will mean $H^*(BG)$-torsion.

**Theorem 1.5** (Borel-type localization formula for a conjugacy class). Let $G$ be a compact connected Lie group acting on a compact space $M$, and $C$ a conjugacy class in $G$. Then the inclusion $i : M^C \to M$ induces an isomorphism in localized equivariant cohomology
\[
i^* : \hat{H}^*_G(M) \to \hat{H}^*_G(M^C).
\]

**Proof.** Let $U$ be a $G$-invariant tubular neighborhoods of $M^C$. Then $\{U, M - M^C\}$ is a $G$-invariant open cover of $M$. Moreover, $H^*_G(U) \simeq H^*_G(M^C)$ because $U$ has the $G$-homotopy type of $M^C$.

By Prop. 1.4, $H^*_G(M - M^C)$ and $H^*_G(U \cap (M - M^C))$ are torsion. Then in the localized Mayer-Vietoris sequence
\[
\cdots \to \hat{H}^{*-1}_G(U \cap (M - M^C)) \to \hat{H}^{*-1}_G(M - M^C) \oplus \hat{H}^{*-1}_G(U) \to \hat{H}^{*-1}_G(U \cap (M - M^C)) \to \cdots,
\]
all the terms except $\hat{H}^*_G(M)$ and $\hat{H}^*_G(U)$ are zero. It follows that
\[
\hat{H}^*_G(M) \to \hat{H}^*_G(U) \simeq \hat{H}^*_G(M^C)
\]
is an isomorphism of $H^*(BG)$-modules. \hfill $\square$

When the group is a torus $T$, a conjugacy class $C$ consist of a single element $t \in T$. If $t$ is generator, it follows that the fixed point set of $t$ is the same as the fixed point set of the whole group $T$, $M^C = M^t = M^T$. Observe that in this case $M^C$ is smooth. Thus Borel’s localization theorem follows from Theorem 1.5 by taking the conjugacy class $C = \{t\}$ in $T$.

2. **The equivariant Euler class**

Suppose a compact connected Lie group $G$ acts on a smooth manifold $M$. Let $C$ be a conjugacy class in $G$, and $M^C$ as before. From now on we assume that $M^C$ is smooth with oriented normal bundle. Denote by $i_M : M^C \to M$ the inclusion map and by $e_M \in H^*_G(M^C)$ the equivariant Euler class of the normal bundle of $M^C$ in $M$. 
Proposition 2.1. Let $M$ be a compact connected oriented $G$-manifold. Then the equivariant Euler class $e_M$ of the normal bundle of $M^C$ in $M$ is invertible in $\hat{H}^*_G(M^C)$.

Proof. Fix a $G$-invariant riemannian metric on $M$. Then the normal bundle $\nu \to M^C$ is a $G$-equivariant vector bundle. Let $\nu_0$ be the normal bundle minus the zero section. Since $\nu_0$ is equivariantly diffeomorphic to an open set in $M - M^C$, by Prop. 1.4 $\hat{H}^*_G(\nu_0)$ vanishes. From the Gysin long exact sequence in localized equivariant cohomology

$$\cdots \to \hat{H}^*_G(\nu_0) \to \hat{H}^*_G(M^C) \times_{e_M} \hat{H}^*_G(M) \to \hat{H}^*_G(\nu_0) \to \cdots$$

it follows that multiplication by the equivariant Euler class gives an automorphism of $\hat{H}^*_G(M^C)$. Thus $e_M$ has an inverse in $\hat{H}^*_G(M^C)$. \hfill \Box

Recall that the inclusion map $i : M^C \to M$ satisfies the identity

$$i^*i_*(x) = xe_M.$$ in equivariant cohomology. From this identity and Prop. 2.1, the map $\varphi : \hat{H}^*_G(M^C) \to \hat{H}^*_G(M)$ given by

$$\varphi(x) = i_* \left( \frac{x}{e_M} \right)$$

is the inverse of the restriction map $i^* : \hat{H}^*_G(M) \to \hat{H}^*_G(M^C)$ of Theorem 1.5.

3. Relative localization formula

Let $N$ be a $G$-manifold, $e_N$ the equivariant Euler class of the normal bundle of $N^C$, and $f : M \to N$ a $G$-equivariant map. There is a commutative diagram of maps

$$\begin{array}{ccc}
M^C & \xrightarrow{i_M} & M \\
\downarrow f^C & & \downarrow f \\
N^C & \xrightarrow{i_N} & N.
\end{array}$$

Let $$(f_C)_* : \hat{H}^*_G(M) \to \hat{H}^*_G(N), \quad (f^C)_* : \hat{H}^*_G(M^C) \to \hat{H}^*_G(N^C)$$
be the push-forward maps in localized equivariant cohomology.

Theorem 3.1 (Relative localization formula). Let $M$ and $N$ be compact oriented manifolds on which a compact connected Lie group $G$ acts, and $f : M \to N$ a $G$-equivariant map. For $a \in \hat{H}^*_G(M)$,

$$(f_G)_* a = (i_N^* a)^{-1} f_C^* \left( \frac{(f_C)_* e_N i_N^* a}{e_M i_M^* a} \right)$$

where the push-forward and restriction maps are in localized equivariant cohomology.

Proof. The commutative diagram (1), induces a commutative diagram in localized equivariant cohomology

$$\begin{array}{ccc}
\hat{H}^*_G(M^C) & \xrightarrow{(f_C)_*} & \hat{H}^*_G(M) \\
\downarrow (f^C)_* & & \downarrow (f_G)_* \\
\hat{H}^*_G(N^C) & \xrightarrow{i_N^*} & \hat{H}^*_G(N).
\end{array}$$
By Prop. 2.1 and the commutativity of the diagram (2),
\[
(f_G)_* a = (f_G)_* i_M^* \left( \frac{1}{e_M} i_M^* a \right) = i_N^* (f^C)_* \left( \frac{1}{e_M} i_M^* a \right).
\]
Hence,
\[
i_N^* (f_G)_* a = i_N^* i_N^* f^C_* \left( \frac{1}{e_M} i_M^* a \right) = e_N^* f^C_* \left( \frac{1}{e_M} i_M^* a \right) = (f^C)_* \left( \frac{(f^C)_* e_N^*}{e_M} i_M^* a \right) \quad \text{(projection formula)}.
\]
By Theorem 1.5, \(i_N^*\) is an isomorphism in localized equivariant cohomology,
\[
(f_G)_* a = (i_N^*)^{-1} (f^C)_* \left( \frac{(f^C)_* e_N^*}{e_M} i_M^* a \right).
\]

Suppose a torus \(T\) acts on compact oriented manifolds \(M\) and \(N\), and \(f : M \to N\) is a \(T\)-equivariant map. The map \(f\) induces a map \(f^T : MT \to NT\) of the fixed point sets. Recall that for a singular element \(t \in T\), \(M^t = MT\). In this case, \(MT\) is always a manifold. Let \(i_M^* : MT \to M\) be the inclusion and \(e_M\) the equivariant Euler class of the normal bundle to \(MT\) in \(M\), and similarly for \(i_N^*\) and \(e_N\).

**Corollary 3.2** (Relative localization formula for a torus action). Let \(M\) and \(N\) be manifolds on which a torus \(T\) acts, and \(f : M \to N\) a \(T\)-equivariant map with compact oriented fibers. For \(a \in H^*_G(M)\),
\[
(f_T)_* a = (i_N^*)^{-1} (f^T)_* \left( \frac{(f^T)_* e_N^*}{e_M} i_M^* a \right),
\]
where the push-forward and restriction maps are in localized equivariant cohomology.

This formula appears in the work of Lian, Liu and Yau in [11].

4. APPLICATIONS TO THE GYSIN HOMOMORPHISM

Let \(G\) be a compact connected Lie group. For a \(G\)-manifold \(M\), let \(h_M : M \to MG\) be the inclusion of \(M\) as a fiber of the bundle \(MG \to BG\) and \(i_M^* : MG \to M\) the inclusion of the fixed point set \(MG\) in \(M\). The map \(h_M\) induces a homomorphism in cohomology
\[
h_M^* : H^*_G(M) \to H^*(M).
\]
The inclusion \(i_M^*\) induces a homomorphism in equivariant cohomology
\[
i_M^* : H^*_G(M) \to H^*_G(MG).
\]

A cohomology class \(a \in H^*(M)\) is said to have an *equivariant extension* \(\tilde{a} \in H^*_G(M)\) under the \(G\) action if under the restriction map \(h_M^* : H^*_G(M) \to H^*(M)\), the equivariant class \(\tilde{a}\) restricts to \(a\).

Suppose \(f : M \to N\) is a \(G\)-equivariant map of compact oriented \(G\)-manifolds. In this section we show that if a class in \(H^*(M)\) has an equivariant extension, then its image under the Gysin map \(f_* : H^*(M) \to H^*(N)\) in ordinary cohomology can be computed from the relative localization formulas (Cor. 3.2 or Th. 3.1).
We consider first the case of an action by a torus $T$. Let $f_T : M_T \to N_T$ be the induced map of homotopy quotients and $f^T : M^T \to N^T$ the induced map of fixed point sets. As before, $e_M$ denotes the equivariant Euler class of the normal bundle of the fixed point set $M^T$ in $M$.

**Proposition 4.1.** Let $f : M \to N$ be a $T$-equivariant map of compact oriented $T$-manifolds. If a cohomology class $a \in H^*(M)$ has an equivariant extension $\tilde{a} \in H^*_T(M)$, then its image under the Gysin map $f_* : H^*(M) \to H^*(N)$ is,

1) in terms of equivariant integration over $M$:

$$f_*a = h^*_N f_T \tilde{a},$$

2) in terms of equivariant integration over the fixed point set $M^T$:

$$f_*a = h^*_N (i^*_N)^{-1} (f^T)_* \left( \frac{(f^T)^* e_N}{e_M} i^*_M \tilde{a} \right).$$

**Proof.** The inclusions $h_M : M \to M_T$ and $h_N : N \to N_T$ fit into a commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{h_M} & M_T \\
\downarrow f & & \downarrow f_T \\
N & \xrightarrow{h_N} & N_T.
\end{array}
$$

This diagram is Cartesian in the sense that $M$ is the inverse image of $N$ under $f_T$. Hence, the push-pull formula $f_* h^*_M = h^*_N f_T$ holds. Then

$$f_* a = f_* h^*_M \tilde{a} = h^*_N f_T \tilde{a}.$$  

2) follows from 1) and the relative localization formula for a torus action (Cor. 3.2).

□

Using the relative localization formula for a conjugacy class, one obtains analogously a push-forward formula in terms of the fixed point sets of a conjugacy class. Now $h_M$ and $i_M$ are the inclusion maps

$$h_M : M \to M_G, \quad i_M : M^C \to M,$$

$e_M$ is the equivariant Euler class of the normal bundle of $M^C$ in $M$, and $f^C : M^C \to N^C$ is the induced map on the fixed point sets of the conjugacy class $C$.

**Proposition 4.2.** Let $f : M \to N$ be a $G$-equivariant map of compact oriented $G$-manifolds. Assume that the fixed point sets $M^C$ and $N^C$ are smooth with oriented normal bundle. For a class $a \in H^*(M)$ that has an equivariant extension $\tilde{a} \in H^*_G(M)$,

$$f_* a = h^*_N (i^*_N)^{-1} (f^C)_* \left( \frac{(f^C)^* e_N}{e_M} i^*_M \tilde{a} \right).$$

5. **Example: the Gysin homomorphism of flag manifolds**

Let $G$ be a compact connected Lie group with maximal torus $T$, and $H$ a closed subgroup of $G$ containing $T$. In [1] Akyildiz and Carrell compute the Gysin homomorphism for the canonical projection $f : G/T \to G/H$. In this section we deduce the formula of Akyildiz and Carrell from the relative localization formula in equivariant cohomology.

Let $N_G(T)$ be the normalizer of the torus $T$ in the group $G$. The Weyl group $W_G$ of $T$ in $G$ is $W_G = N_G(T)/T$. We use the same letter $w$ to denote an element
of the Weyl group $W_G$ and a lift of the element to the normalizer $N_G(T)$. The Weyl group acts on $G/T$ by

$$(gT)w = gwT \quad \text{for } gT \in G/T \text{ and } w \in W_G.$$ 

This induces an action of $W_G$ on the cohomology ring $H^*(G/T)$.

We may also consider the Weyl group $W_H$ of $T$ in $H$. By restriction the Weyl group $W_H$ acts on $G/T$ and on $H^*(G/T)$.

To each character $\gamma$ of $T$ with representation space $\mathbb{C}$, one associates a complex line bundle

$L_\gamma := G \times_T \mathbb{C}_\gamma$

over $G/T$. Fix a set $\Delta^+(H)$ of positive roots for $T$ in $H$, and extend $\Delta^+(H)$ to a set $\Delta^+_{\mathbb{C}}$ of positive roots for $T$ in $G$.

**Theorem 5.1** ([1]). The Gysin homomorphism $f_* : H^*(G/T) \to H^*(G/H)$ is given by, for $a \in H^*(G/T)$,

$$f_*a = \sum_{w \in W_H} (-1)^w w \cdot a \prod_{\alpha \in \Delta^+(H)} c_1(L_{\alpha}).$$

**Remark 5.2.** There are two other ways to obtain this formula. First, using representation theory, Brion [7] proves a push-forward formula for flag bundles that includes Th. 5.1 as a special case. Secondly, since $G/T \to G/H$ is a fiber bundle with equivariantly formal fibers, the method of [14] using the ABBV localization theorem also applies.

To deduce Th. 5.1 from Prop. 4.1 we need to recall a few facts about the cohomology and equivariant cohomology of $G/T$ and $G/H$ (see [13], [14]).

**Cohomology ring of $BT$.** Let $ET \to BT$ be the universal principal $T$-bundle. To each character $\gamma$ of $T$, one associates a complex line bundle $S_\gamma$ over $BT$:

$$S_\gamma := ET \times_T \mathbb{C}_\gamma.$$ 

For definiteness, fix a basis $\chi_1, \ldots, \chi_\ell$ for the character group $\hat{T}$, where we write the characters additively, and set

$$u_i = c_1(S_{\chi_i}) \in H^2(BT), \quad z_i = c_1(L_{\chi_i}) \in H^2(G/T).$$

Let $R = \text{Sym}(\hat{T})$ be the symmetric algebra over $\mathbb{Q}$ generated by $\hat{T}$. The map $\gamma \mapsto c_1(S_{\gamma})$ induces an isomorphism

$$R = \text{Sym}(\hat{T}) \to H^*(BT) = \mathbb{Q}[u_1, \ldots, u_\ell].$$

The map $\gamma \mapsto c_1(L_{\gamma})$ induces an isomorphism

$$R = \text{Sym}(\hat{T}) \to \mathbb{Q}[z_1, \ldots, z_\ell].$$

The Weyl groups $W_G$ and $W_H$ act on the characters of $T$ and hence on $R$: for $w \in W_G$ and $\gamma \in \hat{T}$,

$$(w \cdot \gamma)(t) = \gamma(w^{-1}tw).$$

**Cohomology rings of flag manifolds.** The cohomology rings of $G/T$ and $G/H$ are described in [5]:

$$H^*(G/T) \simeq \frac{R_{W_G}}{(R_{W_G}^+)} \simeq \frac{\mathbb{Q}[z_1, \ldots, z_\ell]}{(\mathbb{R}_{W_G}^+)}.$$

$$H^*(G/H) \simeq \frac{R_{WH}}{(R_{W_G}^+)} \simeq \frac{\mathbb{Q}[z_1, \ldots, z_\ell]_{WH}}{(\mathbb{R}_{W_G}^+)}.$$ 

where $(R_{W_G}^+)$ denotes the ideal generated by the $W_G$-invariant homogeneous polynomials of positive degree.
The torus $T$ acts on $G/T$ and $G/H$ by left multiplication. Their equivariant cohomology rings are (see [6], [13])

$$H^*_T(G/T) = \frac{\mathbb{Q}[u_1, \ldots, u_t, y_1, \ldots, y_d]}{J},$$

$$H^*_T(G/H) = \frac{\mathbb{Q}[u_1, \ldots, u_t] \otimes (\mathbb{Q}[y_1, \ldots, y_d]^{W_H})}{J},$$

where $J$ denotes the ideal generated by $q(y) - q(u)$ for $q \in R^{W_G}$.

### Fixed point sets.

The fixed point sets of the $T$-action on $G/T$ and on $G/H$ are the Weyl group $W_G$ and the coset space $W_G/W_H$ respectively. Since these are finite sets of points,

$$H^*_T(W_G) = \oplus_{w \in W_G} H^*_T(\{w\}) \cong \oplus_{w \in W_G} \mathbb{R},$$

$$H^*_T(W_G/W_H) = \oplus_{w \in W_G/W_H} \mathbb{R}.$$  

Thus, we may view an element of $H^*_T(W_G)$ as a function from $W_G$ to $R$, and an element of $H^*_T(W_G/W_H)$ as a function from $W_G/W_H$ to $R$.

Let $h_M : M \rightarrow M_T$ be the inclusion of $M$ as a fiber in the fiber bundle $M_T \rightarrow BT$ and $i_M : M_T \rightarrow M$ the inclusion of the fixed point set $M^T$ in $M$. Note that $i_M$ is $T$-equivariant and induces a homomorphism in $T$-equivariant cohomology, $i^*_M : H^*_T(M) \rightarrow H^*_T(M^T)$. In order to apply Prop. 4.1, we need to know how to calculate the restriction maps

$$h^*_M : H^*_T(M) \rightarrow H^*(M) \quad \text{and} \quad i^*_M : H^*_T(M) \rightarrow H^*_T(M^T)$$

as well as the equivariant Euler class $e_M$ of the normal bundle to the fixed point set $M^T$, for $M = G/T$ and $G/H$. This is done in [13].

### Restriction and equivariant Euler class formulas for $G/T$.

Since $h^*_M : H^*_T(M) \rightarrow H^*(M)$ is the restriction to a fiber of the bundle $M_T \rightarrow BT$, and the bundle $K_{x_i} = (L_{x_i})_T$ on $M_T$ pulls back to $L_{x_i}$ on $M$,

$$h^*_M(u_i) = 0, \quad h^*_M(y_i) = h^*_M(c_1(K_{x_i})) = c_1(L_{x_i}) = z_i. \quad (3)$$

Let $i_w : \{w\} \rightarrow G/T$ be the inclusion of the fixed point $w \in W_G$ and

$$i^*_w : H^*_T(G/T) \rightarrow H^*_T(\{w\}) = R$$

the induced map in equivariant cohomology. By ([13], Prop. 2), for $p(y) \in H^*_T(G/T)$,

$$i^*_w u_i = u_i, \quad i^*_w p(y) = w \cdot p(u), \quad i^*_w c_1(K_y) = w \cdot c_1(S_y). \quad (4)$$

Thus, the restriction of $p(y)$ to the fixed point set $W_G$ is the function $i^*_M p(y) : W_G \rightarrow R$ whose value at $w \in W_G$ is

$$i^*_M p(y))(w) = w \cdot p(u). \quad (5)$$

The equivariant Euler class of the normal bundle to the fixed point set $W_G$ assigns to each $w \in W_G$ the equivariant Euler class of the normal bundle $\nu_w$ at $w$; thus, it is also a function $e_M : W_G \rightarrow R$. By ([13], Prop. 6),

$$e_M(w) = e^T(\nu_w) = w \left( \prod_{\alpha \in \Delta^+} c_1(S_\alpha) \right) = (-1)^w \prod_{\alpha \in \Delta^+} c_1(S_\alpha). \quad (6)$$
Restriction and equivariant Euler class formulas for $G/H$. For the manifold $M = G/H$, the formulas for the restriction maps $h_N^*$ and $i_N^*$ are the same as in (3) and (4), except that now the polynomial $p(y)$ must be $W_H$-invariant. In particular,

\[(7) \quad h_N^*(u_i) = 0, \quad h_N^*(y) = p(z), \quad h_N^*(e_1(K_\gamma)) = c_1(L_\gamma),\]

and

\[(8) \quad (i_N^*p(y))(wW_H) = w \cdot p(u).\]

If $\gamma_1, \ldots, \gamma_m$ are characters of $T$ such that $p(e_1(K_{\gamma_1}), \ldots, e_1(K_{\gamma_m}))$ is invariant under the Weyl group $W_H$, then

\[(9) \quad (i_N^*p(e_1(K_{\gamma_1}), \ldots, e_1(K_{\gamma_m}))(wW_H) = w \cdot p(e_1(S_{\gamma_1}), \ldots, e_1(S_{\gamma_m})).\]

The equivariant Euler class of the normal bundle of the fixed point set $W_G/W_H$ is the function $e_N : W_G/W_H \rightarrow R$ given by

\[(10) \quad e_N(wW_H) = w \cdot \left( \prod_{\alpha \in \Delta^+ - \Delta^+ (H)} c_1(S_\alpha) \right).\]

Proof of Th. 5.1. With $M = G/T$ and $N = G/H$ in Prop. 4.1, let

\[p(z) \in H^*(G/T) = \mathbb{Q}[z_1, \ldots, z_{\ell}] / (H^*_{W_G}).\]

It is the image of $p(y) \in H^*_T(G/T)$ under the restriction map $h_M^* : H^*_T(G/T) \rightarrow H^*_T(G/T)$. By Prop. 4.1,

\[(11) \quad f_*p(z) = f_*h_M^*p(y) = h_M^*f_T*p(y)\]

and

\[f_T*p(y) = (i_N^*)^{-1}(f^T)_* \left( \frac{(f^T)^*e_N}{e_M} i_M^*p(y) \right).\]

By Eq. (5), (6), and (10), for $w \in W_G$,

\[(i_M^*p(y))(w) = i_M^*p(y) = w \cdot p(u),\]

and

\[\left( \frac{(f^T)^*e_N}{e_M} \right)(w) = \frac{(f^T)^*e_N(wW_H)}{e_M(w)} = w \cdot \left( \frac{\prod_{\alpha \in \Delta^+ - \Delta^+(H)} c_1(S_\alpha)}{\prod_{\alpha \in \Delta^+} c_1(S_\alpha)} \right) \frac{1}{w \cdot \left( \prod_{\alpha \in \Delta^+(H)} c_1(S_\alpha) \right)}.\]

To simplify the notation, define temporarily the function $k : W_G \rightarrow R$ by

\[k(w) = w \cdot \left( \frac{p(u)}{\prod_{\alpha \in \Delta^+(H)} c_1(S_\alpha)} \right).\]

Then

\[(12) \quad f_T*p(y) = (i_M^*)^{-1}(f^T)_*(k).\]

Now $(f^T)_*(k) \in H^*_T(W_G/W_H)$ is the function: $W_G/W_H \rightarrow R$ whose value at the point $wW_H$ is obtained by summing over the fiber of $f^T : W_G \rightarrow W_G/W_H$ above $wW_H$. Hence,

\[((f^T)_*k)(wW_H) = \sum_{wv \in wW_H} wv \cdot \left( \frac{p(u)}{\prod_{\alpha \in \Delta^+(H)} c_1(S_\alpha)} \right) \frac{1}{w \cdot \left( \prod_{\alpha \in \Delta^+(H)} c_1(S_\alpha) \right)}.\]
By (9), the inverse image of this expression under $i_N^*$ is

$$\left( i_N^* \right)^{-1}(f^T)_{*}k = \sum_{v \in W_H} v \cdot \left( \frac{p(y)}{\prod_{\alpha \in \Delta^+(H)} c_1(K_{\alpha})} \right).$$

Finally, combining (11), (12), (13) and (7),

$$f_*p(z) = h_N^*(f_T)^*p(y) = \sum_{v \in W_H} v \cdot \left( \frac{p(z)}{\prod_{\alpha \in \Delta^+(H)} c_1(L_{\alpha})} \right).$$

\[\square\]

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