GENERALIZING THE LOCALIZATION FORMULA IN EQUIVARIANT COHOMOLOGY

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ABSTRACT. We give a generalization of the Atiyah-Bott-Berline-Vergne localization theorem for the equivariant cohomology of a torus action. We replace the manifold having a torus action by an equivariant map of a compact connected Lie group. This provides a systematic method for calculating the Gysin homomorphism in ordinary cohomology of an equivariant map. As an example, we recover a formula of Akyildiz-Carrell for the Gysin homomorphism of flag manifolds.

Suppose M is a compact oriented manifold on which a torus T acts. The Atiyah-Bott-Berline-Vergne localization formula calculates the integral of an equivariant cohomology class on M in terms of an integral over the fixed point set M^T. This formula has found many applications, for example, in analysis, topology, symplectic geometry, and algebraic geometry (see [2], [8], [10], [14]). Similar, but not entirely analogous, formulas exist in K-theory ([3]), cobordism theory ([12]), and algebraic geometry ([9]).

Taking cues from the work of Atiyah and Segal in K-theory [3], we state and prove a localization formula for a compact connected Lie group in terms of the fixed point set of a conjugacy class in the group. As an application, the formula can be used to calculate the Gysin homomorphism in ordinary cohomology of an equivariant map. For a compact connected Lie group G with maximal torus T and a closed subgroup H containing T, we work out as an example the Gysin homomorphism of the canonical projection $f : G/T \longrightarrow G/H$, a formula first obtained by Akyildiz and Carrell [1].

The application to the Gysin map in this article complements that of [14]. The previous article [14] shows how to use the ABBV localization formula to calculate the Gysin map of a fiber bundle. This article shows how to use the relative localization formula to calculate the Gysin map of an equivariant map.

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1. Borel-type localization formula for a conjugacy class

Suppose a compact connected Lie group G acts on a manifold M. For $g \in G$, define M^g to be the fixed point set of g:

$$M^g = \{x \in M \mid g \cdot x = x\}.$$

The set M^g is not G-invariant. The G-invariant subset it generates is

$$\cup_{h \in G} h \cdot (M^g) = \cup_{h \in G} M^{hgh^{-1}} = \cup_{k \in C(g)} M^k$$

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where $C(g)$ is the conjugacy class of g. This suggests that for every conjugacy class C in G, we consider the set M^C of elements of M that are fixed by at least one element of the conjugacy class C:

$$M^C = \cup_{g\in C} M^g.$$

Then M^C is a closed G-subset of M ([3], footnote 1, p. 532); however it is not always smooth. From now on we make the assumption that M^C is smooth.

Remark 1.1. If T is a maximal torus in the compact connected Lie group G and $\dim T = \ell$, then

$$H^*(BG) = H^*(BT)^{W_G} = \mathbb{Q}[u_1, \dots, u_\ell]^{W_G}.$$

Thus, $H^*(BG)$ is an integral domain. Let Q be its field of fractions. For any $H^*(BG)$-module W, we define the localization of W with respect to the zero ideal in $H^*(BG)$ to be

$$\hat{W} := W \otimes_{H^*(BG)} Q.$$

It is easily verified that W is $H^*(BG)$-torsion if and only if $\hat{W} = 0$. For a G-manifold M, we call $\hat{H}^*_G(M)$ the *localized equivariant cohomology* of M.

Proposition 1.2. *Let M be a T-manifold with a finite number of orbit types and no fixed points. Then $H^*_T(M)$ is torsion.*

Proof. For p in M let $\mathfrak{t}_p$ be the Lie algebra of the stabilizer of p. Since the T action on M has finitely many orbit types and has no fixed points, it follows that only a finite number of proper subspaces $\mathfrak{t}_1, \mathfrak{t}_2, \dots \mathfrak{t}_r$ of $\mathfrak{t}$ can occur as $\mathfrak{t}_p$ for p in M.

Let v be a vector in $\mathfrak{t}$ not in the union of the $\mathfrak{t}_i$. Then the induced vector field v_M in M is nowhere zero. Consider the invariant 1-form η on M defined by $\eta(X) = \langle v, X\rangle$. Thus $\eta(v) \equiv 1$ and $d_C(\eta) = d\eta - u_1$. Therefore

$$1 - \frac{d\eta}{u_1} = d_C\left(-\frac{\eta}{u_1}\right)$$

and

$$1 \quad = \quad \left\{ d_C\left(-\frac{\eta}{u_1}\right)\right\} \cdot \left(1 - \frac{d\eta}{u_1}\right)^{-1}.$$

Note that

$$\left(1 - \frac{d\eta}{u_1}\right)^{-1} = \frac{d\eta}{u_1} + \left(\frac{d\eta}{u_1}\right)^2 + \cdots + \left(\frac{d\eta}{u_1}\right)^k.$$

Since $d_C(d\eta) = -u_j\iota_j d\eta = u_j d\iota_j\eta = u_j d(\delta_1^j) = 0$ it follows that

$$1 = d_C\left\{ -\frac{\eta}{u_1} \cdot \left(1 - \frac{d\eta}{u_1}\right)^{-1}\right\}.$$

Hence $H^*_T(M)$ is torsion. □

Lemma 1.3. *If $H^*_T(M)$ is $H^*(BT)$-torsion, then $H^*_G(M)$ is $H^*(BG)$-torsion.*

Proof. Recall that the map $\psi : H^*_G(M) \longrightarrow H^*_T(M)$ is injective. Since $H^*_T(M)$ is $H^*(BT)$-torsion, there is $a \in H^*(BT)$ such that $a \cdot 1_{H^*_T(M)} = 0$. Consider the average of a over the Weyl group W of G with respect to T,

$$\tilde{a} = \frac{1}{|W|}(a + \omega_1 a + \cdots + \omega_r a) \in H^*(BG).$$

Under ψ, the element $\tilde{a}\cdot 1_{H^*_G(M)}$ goes to

$$\frac{1}{|W|}(\omega_1 a + \cdots + \omega_r a)1_{H^*_T(M)}.$$

But $(\omega_j a)1_{H^*_T(M)} = \omega_j(a1_{H^*_T(M)}) = 0$ for any j. Thus $\tilde{a}\cdot 1_{H^*_G(M)} = 0$ in $H^*_G(M)$. $\square$

Proposition 1.4. *Let G be a compact connected Lie group acting on a compact manifold M. Let U be a G-invariant set such that U^C is empty, then the equivariant cohomology $H^*_G(U)$ is $H^*(BG)$-torsion.*

Proof. The set U^C can be described

$$U^C = \{p \in U : hSh^{-1} \subset G_p \text{ for some } h \in G\}$$

where S is the closure of the group generated by some $g \in C$.

Let T a maximal torus containing S. For every $p \in M$, T is not contained in hG_ph^{-1} for all $h \in G$. Therefore the fixed point set of M with respect to T is empty.

Since M is compact, U has only a finite number of orbit types. Thus from Prop. 1.2 and Lemma 1.3 it follows that $H^*_G(U)$ is torsion. $\square$

In the rest of this section, "torsion" will mean $H^*(BG)$-torsion.

Theorem 1.5 (Borel-type localization formula for a conjugacy class)**.** *Let G be a compact connected Lie group acting on a compact space M, and C a conjugacy class in G. Then the inclusion $i : M^C \longrightarrow M$ induces an isomorphism in localized equivariant cohomology*

$$i^* : \hat{H}^*_G(M) \longrightarrow \hat{H}^*_G(M^C).$$

Proof. Let U be a G-invariant tubular neighborhoods of M^C. Then $\{U, M - M^C\}$ is a G-invariant open cover of M. Moreover, $H^*_G(U) \simeq H^*_G(M^C)$ because U has the G-homotopy type of M^C.

By Prop. 1.4, $H^*_G(M - M^C)$ and $H^*_G(U \cap (M - M^C))$ are torsion. Then in the localized Mayer-Vietoris sequence

$$\begin{aligned} &\qquad\qquad\qquad\qquad\qquad\qquad \ldots \longrightarrow \hat{H}^{*-1}_G(U \cap (M - M^C)) \\ &\longrightarrow \hat{H}^*_G(M) \longrightarrow \hat{H}^*_G(M - M^C) \oplus \hat{H}^*_G(U) \longrightarrow \hat{H}^*_G(U \cap (M - M^C)) \longrightarrow \ldots, \end{aligned}$$

all the terms except $\hat{H}^*_G(M)$ and $\hat{H}^*_G(U)$ are zero. It follows that

$$\hat{H}^*_G(M) \longrightarrow \hat{H}^*_G(U) \simeq \hat{H}^*_G(M^C)$$

is an isomorphism of $H^*(BG)$-modules. $\square$

When the group is a torus T, a conjugacy class C consist of a single element $t \in T$. If t is generator, it follows that the fixed point set of t is the same as the fixed point set of the whole group T, $M^C = M^t = M^T$. Observe that in this case M^C is smooth. Thus Borel's localization theorem follows from Theorem 1.5 by taking the conjugacy class $C = \{t\}$ in T.

2. The equivariant Euler class

Suppose a compact connected Lie group G acts on a smooth manifold M. Let C be a conjugacy class in G, and M^C as before. From now on we assume that M^C is smooth with oriented normal bundle. Denote by $i_M : M^C \longrightarrow M$ the inclusion map and by $e_M \in H^*_G(M^C)$ the equivariant Euler class of the normal bundle of M^C in M.

Proposition 2.1. *Let M be a compact connected oriented G-manifold. Then the equivariant Euler class e_M of the normal bundle of M^C in M is invertible in $\hat{H}^*_G(M^C)$.*

Proof. Fix a G-invariant riemannian metric on M. Then the normal bundle $\nu \longrightarrow M^C$ is a G-equivariant vector bundle. Let ν_0 be the normal bundle minus the zero section. Since ν_0 is equivariantly diffeomorphic to an open set in $M - M^C$, by Prop. 1.4 $\hat{H}^*_G(\nu_0)$ vanishes. From the Gysin long exact sequence in localized equivariant cohomology

$$\cdots \longrightarrow \hat{H}^*_G(\nu_0) \longrightarrow \hat{H}^*_G(M^C) \xrightarrow{\times e_M} \hat{H}^*_G(M^C) \longrightarrow \hat{H}^*_G(\nu_0) \longrightarrow \cdots$$

it follows that multiplication by the equivariant Euler class gives an automorphism of $\hat{H}^*_G(M^C)$. Thus e_M has an inverse in $\hat{H}^*_G(M^C)$. □

Recall that the inclusion map $i : M^C \longrightarrow M$ satisfies the identity

$$i^* i_*(x) = x e_M.$$

in equivariant cohomology. From this identity and Prop. 2.1, the map $\varphi : \hat{H}^*_G(M^C) \longrightarrow \hat{H}^*_G(M)$ given by

$$\varphi(x) = i_* \left(\frac{x}{e_M} \right)$$

is the inverse of the restriction map $i^* : \hat{H}^*_G(M) \longrightarrow \hat{H}^*_G(M^C)$ of Theorem 1.5.

3. Relative localization formula

Let N be a G-manifold, e_N the equivariant Euler class of the normal bundle of N^C, and $f : M \longrightarrow N$ a G-equivariant map. There is a commutative diagram of maps

$$\begin{tikzcd} M^C \arrow[r, "i_M"] \arrow[d, "f^C"'] & M \arrow[d, "f"] \\ N^C \arrow[r, "i_N"'] & N. \end{tikzcd} \tag{1}$$

Let

$$(f_G)_* : \hat{H}^*_G(M) \longrightarrow \hat{H}^*_G(N), \qquad f^C_* : \hat{H}^*_G(M^C) \longrightarrow \hat{H}^*_G(N^C)$$

be the push-forward maps in localized equivariant cohomology.

Theorem 3.1 (Relative localization formula)**.** *Let M and N be compact oriented manifolds on which a compact connected Lie group G acts, and $f : M \longrightarrow N$ a G-equivariant map. For $a \in H^*_G(M)$,*

$$(f_G)_* a = (i_N^*)^{-1} f^C_* \left(\frac{(f^C)^* e_N}{e_M} i_M^* a \right)$$

where the push-forward and restriction maps are in localized equivariant cohomology.

Proof. The commutative diagram (1), induces a commutative diagram in localized equivariant cohomology

$$\begin{tikzcd} \hat{H}^*_G(M^C) \arrow[r, "i_{M*}"] \arrow[d, "(f^C)_*"'] & \hat{H}^*_G(M) \arrow[d, "(f_G)_*"] \\ \hat{H}^*_G(N^C) \arrow[r, "i_{N*}"'] & \hat{H}^*_G(N). \end{tikzcd} \tag{2}$$

By Prop. 2.1 and the commutativity of the diagram (2),

$$\begin{aligned}(f_G)_* a &= (f_G)_* i_{M*} \frac{1}{e_M} i_M^* (a)\\ &= i_{N*}(f^C)_* \left(\frac{1}{e_M} i_M^* a\right).\end{aligned}$$

Hence,

$$\begin{aligned} i_N^*(f_G)_* a &= i_N^* i_{N*} f_*^C \left(\frac{1}{e_M} i_M^* a\right)\\ &= e_N f_*^C \left(\frac{1}{e_M} i_M^* a\right)\\ &= (f^C)_* \left(\frac{(f^C)^* e_N}{e_M} i_M^* a\right) \qquad \text{(projection formula).}\end{aligned}$$

By Theorem 1.5, i_N^* is an isomorphism in localized equivariant cohomology,

$$(f_G)_* a = (i_N^*)^{-1} (f^C)_* \left(\frac{(f^C)^* e_N}{e_M} i_M^* a\right).$$

□

Suppose a torus T acts on compact oriented manifolds M and N, and $f : M \longrightarrow N$ is a T-equivariant map. The map f induces a map $f^T : M^T \longrightarrow N^T$ of the fixed point sets. Recall that for a singular element t in T, $M^t = M^T$. In this case, M^T is always a manifold. Let $i_M : M^T \longrightarrow M$ be the inclusion and e_M the equivariant Euler class of the normal bundle to M^T in M, and similarly for i_N and e_N.

Corollary 3.2 (Relative localization formula for a torus action)**.** *Let M and N be manifolds on which a torus T acts, and $f : M \longrightarrow N$ a T-equivariant map with compact oriented fibers. For $a \in \hat{H}_T^*(M)$,*

$$(f_T)_* a = (i_N^*)^{-1} (f^T)_* \left(\frac{(f^T)^* e_N}{e_M} i_M^* a\right),$$

where the push-forward and restriction maps are in localized equivariant cohomology.

This formula appears in the work of Lian, Liu and Yau in [11].

4. Applications to the Gysin homomorphism

Let G be a compact connected Lie group. For a G-manifold M, let $h_M : M \longrightarrow M_G$ be the inclusion of M as a fiber of the bundle $M_G \longrightarrow BG$ and $i_M : M^G \longrightarrow M$ the inclusion of the fixed point set M^G in M. The map h_M induces a homomorphism in cohomology

$$h_M^* : H_G^*(M) \longrightarrow H^*(M).$$

The inclusion i_M induces a homomorphism in equivariant cohomology

$$i_M^* : H_G^*(M) \longrightarrow H_G^*(M^G).$$

A cohomology class $a \in H^*(M)$ is said to have an *equivariant extension* $\tilde{a} \in H_G^*(M)$ under the G action if under the restriction map $h_M^* : H_G^*(M) \longrightarrow H^*(M)$, the equivariant class $\tilde{a}$ restricts to a.

Suppose $f : M \longrightarrow N$ is a G-equivariant map of compact oriented G-manifolds. In this section we show that if a class in $H^*(M)$ has an equivariant extension, then its image under the Gysin map $f_* : H^*(M) \longrightarrow H^*(N)$ in ordinary cohomology can be computed from the relative localization formulas (Cor. 3.2 or Th. 3.1).

We consider first the case of an action by a torus T. Let $f_T : M_T \longrightarrow N_T$ be the induced map of homotopy quotients and $f^T : M^T \longrightarrow N^T$ the induced map of fixed point sets. As before, e_M denotes the equivariant Euler class of the normal bundle of the fixed point set M^T in M.

Proposition 4.1. *Let $f : M \longrightarrow N$ be a T-equivariant map of compact oriented T-manifolds. If a cohomology class $a \in H^*(M)$ has an equivariant extension $\tilde{a} \in H^*_T(M)$, then its image under the Gysin map $f_* : H^*(M) \longrightarrow H^*(N)$ is,*

1) *in terms of equivariant integration over M:*

$$f_* a = h_N^* f_{T*} \tilde{a},$$

2) *in terms of equivariant integration over the fixed point set M^T:*

$$f_* a = h_N^* (i_N^*)^{-1} (f^T)_* \left(\frac{(f^T)^* e_N}{e_M} i_M^* \tilde{a} \right).$$

Proof. The inclusions $h_M : M \longrightarrow M_T$ and $h_N : N \longrightarrow N_T$ fit into a commutative diagram

$$\begin{tikzcd} M \arrow[r, "h_M"] \arrow[d, "f"'] & M_T \arrow[d, "f_T"] \\ N \arrow[r, "h_N"'] & N_T. \end{tikzcd}$$

This diagram is Cartesian in the sense that M is the inverse image of N under f_T. Hence, the push-pull formula $f_* h_M^* = h_N^* f_{T*}$ holds. Then

$$f_* a = f_* h_M^* \tilde{a} = h_N^* f_{T*} \tilde{a}.$$

2) follows from 1) and the relative localization formula for a torus action (Cor. 3.2). □

Using the relative localization formula for a conjugacy class, one obtains analogously a push-forward formula in terms of the fixed point sets of a conjugacy class. Now h_M and i_M are the inclusion maps

$$h_M : M \longrightarrow M_G, \qquad i_M : M^C \longrightarrow M,$$

e_M is the equivariant Euler class of the normal bundle of M^C in M, and $f^C : M^C \longrightarrow N^C$ is the induced map on the fixed point sets of the conjugacy class C.

Proposition 4.2. *Let $f : M \longrightarrow N$ be a G-equivariant map of compact oriented G-manifolds. Assume that the fixed point sets M^C and N^C are smooth with oriented normal bundle. For a class $a \in H^*(M)$ that has an equivariant extension $\tilde{a} \in H^*_G(M)$,*

$$f_* a = h_N^* (i_N^*)^{-1} (f^C)_* \left(\frac{(f^C)^* e_N}{e_M} i_M^* \tilde{a} \right).$$

5. Example: the Gysin homomorphism of flag manifolds

Let G be a compact connected Lie group with maximal torus T, and H a closed subgroup of G containing T. In [1] Akyildiz and Carrell compute the Gysin homomorphism for the canonical projection $f : G/T \longrightarrow G/H$. In this section we deduce the formula of Akyildiz and Carrell from the relative localization formula in equivariant cohomology.

Let $N_G(T)$ be the normalizer of the torus T in the group G. The Weyl group W_G of T in G is $W_G = N_G(T)/T$. We use the same letter w to denote an element of the Weyl group W_G and a lift of the element to the normalizer $N_G(T)$. The Weyl group acts on G/T by

$$(gT)w = gwT \qquad \text{for } gT \in G/T \text{ and } w \in W_G.$$

This induces an action of W_G on the cohomology ring $H^*(G/T)$.

We may also consider the Weyl group W_H of T in H. By restriction the Weyl group W_H acts on G/T and on $H^*(G/T)$.

To each character γ of T with representation space $\mathbb{C}_\gamma$, one associates a complex line bundle

$$L_\gamma := G \times_T \mathbb{C}_\gamma$$

over G/T. Fix a set $\triangle^+(H)$ of positive roots for T in H, and extend $\triangle^+(H)$ to a set $\triangle^+$ of positive roots for T in G.

Theorem 5.1 ([1])**.** *The Gysin homomorphism $f_* : H^*(G/T) \longrightarrow H^*(G/H)$ is given by, for $a \in H^*(G/T)$,*

$$f_*a = \frac{\sum_{w\in W_H}(-1)^w w\cdot a}{\prod_{\alpha\in\triangle^+(H)} c_1(L_\alpha)}.$$

Remark 5.2. There are two other ways to obtain this formula. First, using representation theory, Brion [7] proves a push-forward formula for flag bundles that includes Th. 5.1 as a special case. Secondly, since $G/T \longrightarrow G/H$ is a fiber bundle with equivariantly formal fibers, the method of [14] using the ABBV localization theorem also applies.

To deduce Th. 5.1 from Prop. 4.1 we need to recall a few facts about the cohomology and equivariant cohomology of G/T and G/H (see [13], [14]).

Cohomology ring of BT. Let $ET \longrightarrow BT$ be the universal principal T-bundle. To each character γ of T, one associates a complex line bundle S_γ over BT:

$$S_\gamma := ET \times_T \mathbb{C}_\gamma.$$

For definiteness, fix a basis $\chi_1, \ldots, \chi_\ell$ for the character group $\hat{T}$, where we write the characters additively, and set

$$u_i = c_1(S_{\chi_i}) \in H^2(BT), \qquad z_i = c_1(L_{\chi_i}) \in H^2(G/T).$$

Let $R = \mathrm{Sym}(\hat{T})$ be the symmetric algebra over $\mathbb{Q}$ generated by $\hat{T}$. The map $\gamma \mapsto c_1(S_\gamma)$ induces an isomorphism

$$R = \mathrm{Sym}(\hat{T}) \longrightarrow H^*(BT) = \mathbb{Q}[u_1, \ldots, u_\ell].$$

The map $\gamma \mapsto c_1(L_\gamma)$ induces an isomorphism

$$R = \mathrm{Sym}(\hat{T}) \longrightarrow \mathbb{Q}[z_1, \ldots, z_\ell].$$

The Weyl groups W_G and W_H act on the characters of T and hence on R: for $w \in W_G$ and $\gamma \in \hat{T}$,

$$(w\cdot\gamma)(t) = \gamma(w^{-1}tw).$$

Cohomology rings of flag manifolds. The cohomology rings of G/T and G/H are described in [5]:

$$H^*(G/T) \simeq \frac{R}{(R_+^{W_G})} \simeq \frac{\mathbb{Q}[z_1, \ldots, z_\ell]}{(R_+^{W_G})},$$
$$H^*(G/H) \simeq \frac{R^{W_H}}{(R_+^{W_G})} \simeq \frac{\mathbb{Q}[z_1, \ldots, z_\ell]^{W_H}}{(R_+^{W_G})},$$

where $(R_+^{W_G})$ denotes the ideal generated by the W_G-invariant homogeneous polynomials of positive degree.

The torus T acts on G/T and G/H by left multiplication. Their equivariant cohomology rings are (see [6], [13])

$$H_T^*(G/T) = \frac{\mathbb{Q}[u_1, \dots, u_\ell, y_1, \dots, y_\ell]}{J},$$
$$H_T^*(G/H) = \frac{\mathbb{Q}[u_1, \dots, u_\ell] \otimes (\mathbb{Q}[y_1, \dots, y_\ell]^{W_H})}{J},$$

where J denotes the ideal generated by $q(y) - q(u)$ for $q \in R_+^{W_G}$.

Fixed point sets. The fixed point sets of the T-action on G/T and on G/H are the Weyl group W_G and the coset space W_G/W_H respectively. Since these are finite sets of points,

$$H_T^*(W_G) = \oplus_{w \in W_G} H_T^*(\{w\}) \simeq \oplus_{w \in W_G} R,$$
$$H_T^*(W_G/W_H) = \oplus_{wW_H \in W_G/W_H} R.$$

Thus, we may view an element of $H_T^*(W_G)$ as a function from W_G to R, and an element of $H_T^*(W_G/W_H)$ as a function from W_G/W_H to R.

Let $h_M : M \longrightarrow M_T$ be the inclusion of M as a fiber in the fiber bundle $M_T \longrightarrow BT$ and $i_M : M^T \longrightarrow M$ the inclusion of the fixed point set M^T in M. Note that i_M is T-equivariant and induces a homomorphism in T-equivariant cohomology, $i_M^* : H_T^*(M) \longrightarrow H_T^*(M^T)$. In order to apply Prop. 4.1, we need to know how to calculate the restriction maps

$$h_M^* : H_T^*(M) \longrightarrow H^*(M) \quad \text{and} \quad i_M^* : H_T^*(M) \longrightarrow H_T^*(M^T)$$

as well as the equivariant Euler class e_M of the normal bundle to the fixed point set M^T, for $M = G/T$ and G/H. This is done in [13].

Restriction and equivariant Euler class formulas for G/T. Since $h_M^* : H_T^*(M) \longrightarrow H^*(M)$ is the restriction to a fiber of the bundle $M_T \longrightarrow BT$, and the bundle $K_{\chi_i} = (L_{\chi_i})_T$ on M_T pulls back to L_{χ_i} on M,

$$h_M^*(u_i) = 0, \qquad h_M^*(y_i) = h_M^*(c_1(K_{\chi_i})) = c_1(L_{\chi_i}) = z_i. \tag{3}$$

Let $i_w : \{w\} \longrightarrow G/T$ be the inclusion of the fixed point $w \in W_G$ and

$$i_w^* : H_T^*(G/T) \longrightarrow H_T^*(\{w\}) = R$$

the induced map in equivariant cohomology. By ([13], Prop. 2), for $p(y) \in H_T^*(G/T)$,

$$i_w^* u_i = u_i, \quad i_w^* p(y) = w \cdot p(u), \quad i_w^* c_1(K_\gamma) = w \cdot c_1(S_\gamma). \tag{4}$$

Thus, the restriction of $p(y)$ to the fixed point set W_G is the function $i_M^* p(y) : W_G \longrightarrow R$ whose value at $w \in W_G$ is

$$(i_M^* p(y))(w) = w \cdot p(u). \tag{5}$$

The equivariant Euler class of the normal bundle to the fixed point set W_G assigns to each $w \in W_G$ the equivariant Euler class of the normal bundle ν_w at w; thus, it is also a function $e_M : W_G \longrightarrow R$. By ([13], Prop. 6),

$$e_M(w) = e^T(\nu_w) = w \left(\prod_{\alpha \in \triangle^+} c_1(S_\alpha) \right) = (-1)^w \prod_{\alpha \in \triangle^+} c_1(S_\alpha). \tag{6}$$

Restriction and equivariant Euler class formulas for G/H. For the manifold $M = G/H$, the formulas for the restriction maps h_N^* and i_N^* are the same as in (3) and (4), except that now the polynomial $p(y)$ must be W_H-invariant. In particular,

$$h_N^*(u_i) = 0, \quad h_N^* p(y) = p(z), \quad h_N^*(c_1(K_\gamma)) = c_1(L_\gamma), \tag{7}$$

and

$$(i_N^* p(y))(wW_H) = w \cdot p(u). \tag{8}$$

If $\gamma_1, \ldots, \gamma_m$ are characters of T such that $p(c_1(K_{\gamma_1}), \ldots, c_1(K_{\gamma_m}))$ is invariant under the Weyl group W_H, then

$$(i_N^* p(c_1(K_{\gamma_1}), \ldots, c_1(K_{\gamma_m})))(wW_H) = w \cdot p(c_1(S_{\gamma_1}), \ldots, c_1(S_{\gamma_m})). \tag{9}$$

The equivariant Euler class of the normal bundle of the fixed point set W_G/W_H is the function $e_N : W_G/W_H \longrightarrow R$ given by

$$e_N(wW_H) = w \cdot \left(\prod_{\alpha \in \triangle^+ - \triangle^+(H)} c_1(S_\alpha) \right). \tag{10}$$

Proof of Th. 5.1. With $M = G/T$ and $N = G/H$ in Prop. 4.1, let

$$p(z) \in H^*(G/T) = \mathbb{Q}[z_1, \ldots, z_\ell]/(R_+^{W_G}).$$

It is the image of $p(y) \in H_T^*(G/T)$ under the restriction map $h_M^* : H_T^*(G/T) \longrightarrow H^*(G/T)$. By Prop. 4.1,

$$f_* p(z) = f_* h_M^* p(y) = h_N^* f_{T*} p(y) \tag{11}$$

and

$$f_{T*} p(y) = (i_N^*)^{-1} (f^T)_* \left(\frac{(f^T)^* e_N}{e_M} i_M^* p(y) \right).$$

By Eq. (5), (6), and (10), for $w \in W_G$,

$$(i_M^* p(y))(w) = i_w^* p(y) = w \cdot p(u),$$

and

$$\left(\frac{(f^T)^* e_N}{e_M} \right)(w) = \frac{(f^T)^*(e_N(wW_H))}{e_M(w)} = w \cdot \left(\frac{\prod_{\alpha \in \triangle^+ - \triangle^+(H)} c_1(S_\alpha)}{\prod_{\alpha \in \triangle^+} c_1(S_\alpha)} \right)$$
$$= \frac{1}{w \cdot \left(\prod_{\alpha \in \triangle^+(H)} c_1(S_\alpha) \right)}.$$

To simplify the notation, define temporarily the function $k : W_G \longrightarrow R$ by

$$k(w) = w \cdot \left(\frac{p(u)}{\prod_{\alpha \in \triangle^+(H)} c_1(S_\alpha)} \right).$$

Then

$$f_{T*} p(y) = (i_N^*)^{-1} (f^T)_*(k). \tag{12}$$

Now $(f^T)_*(k) \in H_T^*(W_G/W_H)$ is the function: $W_G/W_H \longrightarrow R$ whose value at the point wW_H is obtained by summing over the fiber of $f^T : W_G \longrightarrow W_G/W_H$ above wW_H. Hence,

$$((f^T)_* k)(wW_H) = \sum_{wv \in wW_H} wv \cdot \left(\frac{p(u)}{\prod_{\alpha \in \triangle^+(H)} c_1(S_\alpha)} \right)$$
$$= w \cdot \sum_{v \in W_H} v \cdot \left(\frac{p(u)}{\prod_{\alpha \in \triangle^+(H)} c_1(S_\alpha)} \right).$$

By (9), the inverse image of this expression under i_N^* is

$$(i_N^*)^{-1}(f^T)_* k = \sum_{v \in W_H} v \cdot \left(\frac{p(y)}{\prod_{\alpha \in \triangle^+(H)} c_1(K_\alpha)} \right). \tag{13}$$

Finally, combining (11), (12), (13) and (7),

$$f_* p(z) = h_N^* (f_T)_* p(y) = \sum_{v \in W_H} v \cdot \left(\frac{p(z)}{\prod_{\alpha \in \triangle^+(H)} c_1(L_\alpha)} \right).$$

□

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