SEIDEL'S REPRESENTATION ON THE HAMILTONIAN GROUP OF A CARTESIAN PRODUCT

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Abstract. Let (M,ω) be a closed symplectic manifold and $\mathrm{Ham}(M,\omega)$ the group of Hamiltonian diffeomorphisms of (M,ω). Then the Seidel homomorphism is a map from the fundamental group of $\mathrm{Ham}(M,\omega)$ to the quantum homology ring $QH_*(M;\Lambda)$. Using this homomorphism we give a sufficient condition for when a nontrivial loop ψ in $\mathrm{Ham}(M,\omega)$ determines a nontrivial loop $\psi\times\mathrm{id}_N$ in $\mathrm{Ham}(M\times N,\omega\oplus\eta)$, where (N,η) is a closed symplectic manifold such that $\pi_2(N)=0$.

1. Introduction

Let (M,ω) be a closed symplectic manifold and $\psi=\{\psi_t\}_{0\leq t\leq 1}$ a loop about the indentity map in the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M,\omega)$. Then associated to ψ, there is a fibration $\pi : P_\psi \longrightarrow S^2$ with fiber M. In [7], P. Seidel defined a group homomorphism $\mathcal{S} : \pi_1(\mathrm{Ham}(M,\omega)) \longrightarrow QH_*(M;\Lambda)$, where $QH_*(M;\Lambda)$ is the quantum homology ring of (M,ω). The map $\mathcal{S}$ is usually called Seidel's representation, since its image lies in the subring of units of $QH_*(M;\Lambda)$, which in turn defines a homomorphism of the quantum homology ring via quantum multiplication. The homomorphism $\mathcal{S}$ can be thought as the quantum analog of Weinstein's action $\mathcal{A} : \pi_1(\mathrm{Ham}(M,\omega)) \longrightarrow \mathbb{R}/P_\omega$ of [8]. The element $\mathcal{S}(\psi)$, is defined in terms of Gromov-Witten invariants related to the moduli space of holomorphic sections of the induced fibration $\pi : P_\psi \longrightarrow S^2$. This homomorphism was used by P. Seidel to detect nontrivial loops in the group $\mathrm{Ham}(M,\omega)$. Further, the type of Gromov-Witten invariants involved in the definition of $\mathcal{S}$, where studied by D. McDuff in [3], to show that the rational cohomology of a Hamiltonian fibration splits.

For very special symplectic manifolds the fundamental group of $\mathrm{Ham}(M,\omega)$ is completely known. The easiest case is when M has dimension 2, since in this case symplectic diffeomorphisms agree with volume preserving diffeomorphisms. Hence the fundamental group of $\mathrm{Ham}(S^2,\omega)$ is isomorphic to $\mathbb{Z}_2$; and $\mathrm{Ham}(\Sigma_g,\omega)$ is contractible for $g\geq 1$. For further details see [6]. In higher dimensions M. Gromov showed in [1] that the fundamental group of $\mathrm{Ham}(\mathbb{C}P^2,\omega_{FS})$ is isomorphic to $\mathbb{Z}_3$ and the fundamental group of $\mathrm{Ham}(S^2\times S^2,\omega\oplus\omega)$ is isomorphic to a semidirect product of $\mathbb{Z}_2$ with itself. The last case is more interesting when the standard symplectic $\omega\oplus\omega$ on $S^2\times S^2$ is replaced by the symplectic form $\lambda\omega\oplus\omega$, where λ is a real constant greater than 1. In this case D. McDuff proved that the fundamental group of $\mathrm{Ham}(S^2\times S^2,\omega\oplus\lambda\omega)$ contains an element of infinite order.

Consider (M,ω) and (N,η) closed symplectic manifolds. Moreover assume that both manifolds are monotone. If ψ is a loop of Hamiltonian diffeomorphisms of

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(M,ω) based at the identity map, then $\psi\times \mathrm{id}_N$ is a loop of Hamiltonian diffeomorphisms of the symplectic manifold $(M\times N,\omega\oplus\eta)$. Then both loops ψ and $\psi\times\mathrm{id}_N$ induce fibrations $\pi : P_\psi \longrightarrow S^2$ and $\pi_1 : P_{\psi\times \mathrm{id}_N} \longrightarrow S^2$ with fibers the symplectic manifolds M and $M\times N$ respectively. This article aims on relating Seidel's representations over $\pi_1(\mathrm{Ham}(M,\omega))$ and $\pi_1(\mathrm{Ham}(M\times N,\omega\oplus\eta))$. We show that Seidel's representation on $\pi_1(\mathrm{Ham}(M\times N,\omega\oplus\eta))$ restricted to elements of the form $\psi\times\mathrm{id}_N$ is essentially the same as Seidel's representation on $\pi_1(\mathrm{Ham}(M,\omega))$.

We achieve this by relating the fundamental groups of $\mathrm{Ham}(M,\omega)$ and $\mathrm{Ham}(M\times N,\omega\oplus\eta)$ and the quantum homology rings of M and $M\times N$. For, let

$$\tau : \pi_1(\mathrm{Ham}(M,\omega)) \longrightarrow \pi_1(\mathrm{Ham}(M\times N,\omega\oplus\eta))$$

be the group homomorphism defined by $\tau(\psi)=\psi\times\mathrm{id}_N$, and

$$\kappa : QH_*(M;\Lambda) \longrightarrow QH_{*+\dim(N)}(M\times N;\Lambda)$$

be the map defined on homogeneous elements by $\kappa(\alpha\otimes q^st^r) = (\alpha\otimes[N])\otimes q^st^r$ where $\alpha\in H_*(M)$ and $\alpha\otimes[N]\in H_*(M\times N)$. Extend Λ-linearly the map κ to all the quantum homology ring $QH_*(M;\Lambda)$. In Section 4 we show that κ is in fact a ring homomorphism under the quantum product, if both manifolds (M,ω) and (N,η) are monotone with the same constant. This statement, as others in this article, is a direct consequence of the fact that the quantum homology ring $QH_*(M\times N;\Lambda)$ satisfies the Künneth formula.

Theorem 1.1. *Let (M,ω) and (N,η) be closed symplectic manifolds. Assume that (M,ω) has dimension $2n$ and is monotone, and that $\pi_2(N)=0$. Then $\mathcal{S}\circ\tau(\psi) = \kappa\circ\mathcal{S}(\psi)$ for every ψ in $\pi_1(\mathrm{Ham}(M,\omega))$. That is the following diagram commutes*

$$\begin{array}{ccc} \pi_1(\mathrm{Ham}(M,\omega)) & \xrightarrow{\ \tau\ } & \pi_1(\mathrm{Ham}(M\times N,\omega\oplus\eta)) \\ \mathcal{S}\downarrow & & \downarrow\mathcal{S} \\ QH_{2n}(M;\Lambda) & \xrightarrow{\ \kappa\ } & QH_{2n+\dim(N)}(M\times N;\Lambda) \end{array}$$

It is important to relate Thm. 1.1 with a result of D. McDuff and S. Tolman. In [5], McDuff and Tolman found a formula for $\mathcal{S}(\psi)$ in the case when ψ is a Hamiltonian circle action on (M,ω). Thus if ψ is a Hamiltonian circle action on (M,ω), denote by $K : M\longrightarrow\mathbb{R}$ the normalized moment map of the circle action and by K_0 the maximum value of K. Let $M_{\max}$ be the symplectic submanifold on which the moment map K achieves its maximun. Note that $M_{\max}$ is part of the fixed point set of the circle action. Finally assume that there is a neighborhood U of $M_{\max}$ such that the action of the circle is free on $U-M_{\max}$. Under the above assumptions McDuff-Tolman formula for the circle action ψ reads

$$\mathcal{S}(\psi) = [M_{\max}]\otimes q^{\mathrm{codim}(M_{\max})/2}t^{-K_0} + \sum_{\{A:\tilde\omega(A)>K_0\}} \alpha_A\otimes q^{-c(A)}t^{-\tilde\omega(A)}. \tag{1}$$

In order to make a clear statement of the goal of this article we postpone to Section 3, the definitions of the elements $A, c(A)$ and $\tilde\omega(A)$ that appear in Eq. (1). Also a word of warning about Eq. (1). We have used the notation of [4] in stating McDuff-Tolman formula and not that of the original paper [5]. At then end of Section 3, we clarify how they are related.

Now let (M, ω) and ψ be as above and (N, η) any closed symplectic manifold. Then $\psi \times \mathrm{id}_N$ is also a Hamiltonian circle action on $M \times N$ with moment map $H : M \times N \longrightarrow \mathbb{R}$ given by $H(p, q) = K(p)$. So defined, the moment map H is normalized. Also $(M \times N)_{\max} = M_{\max} \times N$ and $H_0 = K_0$. Observe that $\mathrm{codim}_M(M_{\max}) = \mathrm{codim}_{M\times N}(M_{\max} \times N)$. Thus McDuff-Tolman formula for the circle action $\psi \times \mathrm{id}_N$ is,

$$\mathcal{S}(\psi \times \mathrm{id}_N) = [M_{\max} \times N] \otimes q^{\mathrm{codim}(M_{\max})/2} t^{-K_0} + \sum_{\{A' : \tilde{\omega}'(A') > K_0\}} \alpha'_{A'} \otimes q^{-c'(A')} t^{-\tilde{\omega}'(A')}. \tag{2}$$

At this point is important to observe that the first term on the right hand side of Eqs. (1) and (2) only differ by the class $[N]$. Well Thm. 1.1 guarantees that if $\pi_2(N) = 0$ not only the first terms of $\mathcal{S}(\psi)$ and $\mathcal{S}(\psi \times \mathrm{id}_N)$ differ by $[N]$, but the equality $\mathcal{S}(\psi) \otimes [N] = \mathcal{S}(\psi \times \mathrm{id}_N)$ holds in $QH_*(M \times N; \Lambda)$ for any loop ψ, not just a circle action.

Notice that the map $\kappa : QH_*(M; \Lambda) \longrightarrow QH_{*+\dim(N)}(M \times N; \Lambda)$ so defined is injective. Therefore Thm. 1.1 tells us when a nontrivial $\psi \in \pi_1(\mathrm{Ham}(M, \omega))$ induces a nontrivial element $\psi \times \mathrm{id}_N$ in $\pi_1(\mathrm{Ham}(M \times N, \omega \oplus \eta))$.

Corollary 1.2. *Let (M, ω) and (N, η) be as in Thm. 1.1. Then if $\psi \in \pi_1(\mathrm{Ham}(M, \omega))$ is such that $\mathcal{S}(\psi) \neq 1 = [M]$, then the loop $\psi \times \mathrm{id}_N$ is also nontrivial in $\pi_1(\mathrm{Ham}(M \times N, \omega \oplus \eta))$.*

Hence if the Seidel representation on $\pi_1(\mathrm{Ham}(M, \omega))$ is injective, we conclude that the group homomorphism $\tau : \pi_1(\mathrm{Ham}(M, \omega)) \longrightarrow \pi_1(\mathrm{Ham}(M \times N, \omega \oplus \eta))$ is also injective. For instance, by the result of McDuff and Tolman we know that the Seidel representation is injective in the case when $M = S^2$ or $\mathbb{C}P^2$. Therefore for any closed symplectic manifold (N, η) such that $\pi_2(N) = 0$, we have that the group homomorphism $\tau : \pi_1(\mathrm{Ham}(M, \omega)) \longrightarrow \pi_1(\mathrm{Ham}(M \times N, \omega \oplus \eta))$ is injective for $M = S^2$ or $\mathbb{C}P^2$.

Example. Let (M, ω) and (N, η) be symplectic manifolds as in Thm. 1.1. Moreover assume that there is a loop γ in $\mathrm{Ham}(M, \omega)$ such that $\mathcal{S}(\gamma)$ has infinite order in $QH_*(M; \Lambda)$ under quantum multiplication. Thus the loop γ also has infinite order in the fundamental group of $\mathrm{Ham}(M, \omega)$.

Hence we have that $\mathcal{S}(\gamma^m)$ is not equal to the identity $1 = [M]$ in $QH_*(M; \Lambda)$ for all $m \in \mathbb{Z}$ different from zero. Then by Cor. 1.2, it follows that the loop $\gamma^m \times \mathrm{id}_N$ is not homologous to the constant loop in $\mathrm{Ham}(M \times N, \omega \oplus \eta)$ for all nonzero m. That is, $\gamma^m \times \mathrm{id}_N$ is an element of infinite order in $\pi_1(\mathrm{Ham}(M \times N, \omega \oplus \eta))$.

Now consider the case when $M = N$. Hence assume that (M, ω) is a closed symplectic manifold such that $\pi_2(M)$ is trivial. Thus (M, ω) is monotone. We are intrested in understanding when a nontrivial loop ψ in $\mathrm{Ham}(M, \omega)$ induces a nontrivial loop $\psi \times \psi$ in $\mathrm{Ham}(M \times M, \omega \oplus \omega)$. That is, we are intrested in the image of the group homomorphism

$$\tau' : \pi_1(\mathrm{Ham}(M, \omega)) \longrightarrow \pi_1(\mathrm{Ham}(M \times M, \omega \oplus \omega))$$

defined by $\tau'(\psi) = \psi \times \psi$.

Consider the map

$$\kappa' : QH_*(M; \Lambda) \longrightarrow QH_{*+*}(M \times M; \Lambda) \simeq (H_*(M) \otimes_{\mathbb{Z}} H_*(M)) \otimes_{\mathbb{Z}} \Lambda$$

defined on homogeneous elements by $\kappa'(\alpha \otimes q^s t^r) = (\alpha \otimes \alpha) \otimes q^{2s} t^{2r}$ where $\alpha \in H_*(M)$. In Section 4 we will review the fact that the Künneth formula holds in quantum homology. Hence the map κ' corresponds to the diagonal map

$$\Delta : QH_*(M;\Lambda) \longrightarrow QH_*(M;\Lambda) \otimes_\Lambda QH_*(M;\Lambda) \simeq QH_*(M \times M;\Lambda)$$

defined by $\Delta(x) = x \otimes x$ for all $x \in QH_*(M;\Lambda)$ via the quantum Künneth formula.

Theorem 1.3. *Let (M,ω) be closed symplectic manifold of dimension $2n$ such that $\pi_2(M)$ is trivial. Then $\mathcal{S} \circ \tau'(\psi) = \kappa' \circ \mathcal{S}(\psi)$ for every ψ in $\pi_1(\mathrm{Ham}(M,\omega))$. That is the following diagram commutes*

$$\begin{array}{ccc} \pi_1(\mathrm{Ham}(M,\omega)) & \xrightarrow{\ \tau'\ } & \pi_1(\mathrm{Ham}(M \times M, \omega \oplus \omega)) \\ \mathcal{S} \downarrow & & \downarrow \mathcal{S} \\ QH_{2n}(M;\Lambda) & \xrightarrow{\ \kappa'\ } & QH_{4n}(M \times M;\Lambda) \end{array}$$

As in the case of Thm. 1.1, one can use McDuff-Tolman formula to verify that Thm. 1.3 works in the case when the Hamiltonian loop ψ is a Hamiltonian circle action. In fact, if (M,ω), ψ, and $K : M \longrightarrow \mathbb{R}$ are as above, that is ψ is a Hamiltonian circle action with normalized moment map K, then $\psi \times \psi$ is a Hamiltonian circle action on the product manifold $(M \times M, \omega \oplus \omega)$ with normalized moment map $H : M \times M \longrightarrow \mathbb{R}$ given by $H(p_1,p_2) = K(p_1) + K(p_2)$. Hence the maximun value H_0 of the moment map H satisfies the relation $H_0 = 2K_0$ and also $(M \times M)_{\max} = M_{\max} \times M_{\max}$. Then in this case McDuff-Tolman formula for the Hamiltonian circle action $\psi \times \psi$ on $M \times M$ is given by

$$\begin{aligned} \mathcal{S}(\psi \times \psi) &= [M_{\max} \times M_{\max}] \otimes q^{\operatorname{codim}_{M \times M}(M_{\max} \times M_{\max})/2} t^{-H_0} + \\ &\qquad \sum_{\{A : \tilde{\omega}(A) > H_0\}} \alpha_A \otimes q^{-c(A)} t^{-\tilde{\omega}(A)} \\ &= [M_{\max} \times M_{\max}] \otimes q^{\operatorname{codim}_M(M_{\max})} t^{-2K_0} + \\ &\qquad \sum_{\{A : \tilde{\omega}(A) > 2K_0\}} \alpha_A \otimes q^{-c(A)} t^{-\tilde{\omega}(A)}. \end{aligned} \tag{3}$$

Comparing the first term on the right hand side of Eqs. (1) and (3), one checks that they are related by the map κ'. Well according to Thm. 1.3, $\mathcal{S}(\psi)$ and $\mathcal{S}(\psi \times \psi)$ are related by the map κ' for any loop ψ in $\mathrm{Ham}(M,\omega)$, not just a Hamiltonian circle action.

As before the map κ' so defined is injective. Hence we have a criteria to determine when the loop $\psi \times \psi$ is nontrivial in $\pi_1(\mathrm{Ham}(M \times M, \omega \oplus \omega))$.

Corollary 1.4. *Let (M,ω) be a closed symplectic manifold such that $\pi_2(M) = 0$. If $\psi \in \pi_1(\mathrm{Ham}(M,\omega))$ is such that $\mathcal{S}(\psi) \neq 1 = [M]$, then the Hamiltonian loop $\psi \times \psi$ is nontrivial in $\pi_1(\mathrm{Ham}(M \times M, \omega \oplus \omega))$.*

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2. Hamiltonian fibrations

Consider (M, ω) a closed symplectic manifold. Let $\psi = \{\psi_t\}_{0 \leq t \leq 1}$ be a loop about the indentity map in the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$. Associated to ψ there is a smooth fibration $\pi : P_\psi \longrightarrow S^2$ with fiber M defined as follows. Let $D^+ = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the closed unit disk with the positive orientation. Then the total space of the fibration P_ψ is defined as

$$(D^+ \times M) \amalg (D^- \times M)/\sim$$

where $(e^{it}, p)_+ \sim (e^{-it}, \psi_t^{-1}(p))_-$ and D^- stands for D^+ with the opposite orientation. This fibration has $\mathrm{Ham}(M, \omega)$ as structure group, and is called **Hamiltonian fibration**. See ([4], p. 251).

In fact there is a one-to-one correspondence between homotopic loops in $\mathrm{Ham}(M, \omega)$ based at the identity map id_M and isomorphic fibrations over S^2 with fiber M and structure group $\mathrm{Ham}(M, \omega)$. In order to avoid cumbersome notation we will use the same notation to denote loops based at the identity in $\mathrm{Ham}(M, \omega)$ and its homotopy class in $\pi_1(\mathrm{Ham}(M, \omega))$, namely $\psi = \{\psi_t\}_{0 \leq t \leq 1}$.

In a Hamiltonian fibration $\pi : P_\psi \longrightarrow S^2$ with fiber the symplectic manifold (M, ω), there exists a closed 2-form $\tilde{\omega}$ on P_ψ such that it restricts to ω_z on every fiber $(P_\psi)_z$ for $z \in S^2$, and such that

$$\pi_!(\tilde{\omega}^{n+1}) = 0,$$

where $\pi_! : H^*(P_\psi) \longrightarrow H^{*-\dim(\mathrm{M})}(S^2)$ stands for integration along the fiber M. A 2-form $\tilde{\omega}$ that satisfies the above conditions is called a **coupling form**. See [2] for more details. The coupling form $\tilde{\omega}$ defines a connection on the fibration, where the horizontal distribution is defined as the $\tilde{\omega}$-complement of the vertical subspace. That is, for $p \in P_\psi$,

$$\mathrm{hor}(T_p P_\psi) = \{v \in T_p P_\psi \mid \tilde{\omega}(v, u) = 0 \text{ for all } u \in \ker(\pi_{*,p})\}.$$

There is another canonical class associated to the fibration $\pi : P_\psi \longrightarrow S^2$, apart from the cohomology class determined by coupling form. Recall that the vertical vector bundle of a fibration is the vector bundle

$$T^{\mathrm{V}} P_\psi = \{v \in T_p P_\psi \mid \pi_{*,p}(v) = 0\}$$

over the total space P_ψ. The coupling form $\tilde{\omega}$ restricted to this subbundle is nondegenerate. Thus the first Chern class of $T^{\mathrm{V}} P_\psi$ is well defined. This class is denoted by $c_\psi := c_1(T^{\mathrm{V}} P_\psi) \in H^2(P_\psi; \mathbb{Z})$.

Let (M, ω) and (N, η) be closed symplectic manifolds, then $(M \times N, \omega \oplus \eta)$ is also a closed symplectic manifold. Note that the 2-form $\omega \oplus \eta$ is a shorthand notation for the 2-form $(\mathrm{pr}_M)^*(\omega) + (\mathrm{pr}_N)^*(\eta)$ on $M \times N$, where pr_M and pr_N are the projection maps from $M \times N$ to M and N respectively. If ψ is a loop in the group $\mathrm{Ham}(M, \omega)$, then $\psi \times \mathrm{id}_N$ is also a loop of Hamiltonian diffeomorphisms of the symplectic manifold $(M \times N, \omega \oplus \eta)$. Thus there is a Hamiltonian fibration $\pi_1 : P_{\psi \times \mathrm{id}_N} \longrightarrow S^2$ with fiber $M \times N$. As before we have the fiber bundle $\pi : P_\psi \longrightarrow S^2$ with fiber M. Define the fibration $\pi_0 : P_\psi \times N \longrightarrow S^2$ where the projection map is defined as $\pi_0(x, q) = \pi(x)$. So defined π_0 is a fiber bundle with fiber $M \times N$. Well both fiber bundles $P_{\psi \times \mathrm{id}_N}$ and $P_\psi \times N$ are isomorphic.

Lemma 2.5. *The fiber bundles $\pi_0 : P_\psi \times N \longrightarrow S^2$ and $\pi_1 : P_{\psi \times \mathrm{id}_N} \longrightarrow S^2$ with fiber $M \times N$ are isomorphic fibrations.*

Proof. Consider the map $\rho : ((D^+ \times M) \amalg (D^- \times M)) \times N \longrightarrow P_{\psi\times \mathrm{id}_N}$ defined as $\rho((u,p),q) = [u,p,q]$. Then

$$\rho((e^{it},p),q) = [e^{it},p,q] = [e^{-it},\psi_t^{-1}(p),\mathrm{id}_N(q)] = \rho((e^{-it},\psi_t^{-1}(p)),q).$$

This means that ρ induces a map on the quotient $P_\psi \times N$. We denote such map by the same letter ρ. So defined $\rho : P_\psi \times N \longrightarrow P_{\psi\times \mathrm{id}_N}$ is smooth and bijective. Moreover for any $([u,p],q) \in P_\psi \times N$ we have that

$$\pi_0([u,p],q) = \pi([u,p]) = u \qquad \text{and} \qquad \pi_1 \circ \rho([u,p],q) = \pi_1([u,p,q]) = u.$$

Therefore $\pi_0 = \pi_1 \circ \rho$ and ρ is fiberwise preserving. That is the fiber bundles $P_\psi \times N$ and $P_{\psi\times \mathrm{id}_N}$ are isomorphic fibrations. □

Thus there are two isomorphic fibrations over S^2 with fiber $M \times N$; $P_{\psi\times \mathrm{id}_N}$ and $P_\psi \times N$. Hence both vertical bundles are isomorphic $T^{\mathrm{V}}(P_{\psi\times \mathrm{id}_N}) \simeq T^{\mathrm{V}}(P_\psi \times N)$ as vector bundles. In order to compare the first Chern classes of both fibrations, consider the projections maps $\lambda_1 : P_\psi \times N \longrightarrow P_\psi$, $\lambda_2 : P_\psi \times N \longrightarrow N$ and the diagram

$$\begin{array}{ccccc}
T^{\mathrm{V}}P_\psi & \longleftarrow & (\lambda_1)^*(T^{\mathrm{V}}P_\psi) \oplus (\lambda_2)^*(TN) & \longrightarrow & TN \\
\downarrow & & \downarrow & & \downarrow \\
P_\psi & \xleftarrow{\lambda_1} & P_\psi \times N & \xrightarrow{\lambda_2} & N
\end{array}$$

where $(\lambda_1)^*(T^{\mathrm{V}}P_\psi)$ and $(\lambda_2)^*(TN)$ stand for the pullback bundles.

Proposition 2.6. *The vector bundles $T^{\mathrm{V}}(P_\psi \times N)$ and $(\lambda_1)^*(T^{\mathrm{V}}P_\psi) \oplus (\lambda_2)^*(TN)$ are isomorphic vector bundles over $P_\psi \times N$.*

Proof. Let $x = (p,q) \in P_\psi \times N$ and $u = u_1 + u_2 \in T_x(P_\psi \times N) \simeq T_pP_\psi \oplus T_qN$. Thus the vector u belongs to $T^{\mathrm{V}}(P_\psi \times N)$ if and only if $(\pi_0)_*(u) = 0$. By the definition of the map π_0, we have $(\pi_0)_{*,x}(u_1+u_2) = (\pi)_{*,p}(u_1) = 0$. Thus $u = u_1 + u_2$ is in $T^{\mathrm{V}}(P_\psi \times N)$ if and only if u_1 is in $T^{\mathrm{V}}P_\psi$ and u_2 is in TN. Therefore

$$T^{\mathrm{V}}(P_\psi \times N) \simeq (\lambda_1)^*(T^{\mathrm{V}}P_\psi) \oplus (\lambda_2)^*(TN)$$

as vector bundles. □

Let $c_\psi \in H^2(P_\psi;\mathbb{Z})$ be the first Chern class of the vector bundle $T^{\mathrm{V}}P_\psi \longrightarrow P_\psi$. And respectively $c_{\psi\times \mathrm{id}_N} \in H^2(P_{\psi\times \mathrm{id}_N};\mathbb{Z}) = H^2(P_\psi \times N;\mathbb{Z})$.

Lemma 2.7. *On $H^2(P_{\psi\times \mathrm{id}_N};\mathbb{Z})$ we have the identity*

$$c_{\psi\times \mathrm{id}_N} = (\lambda_1)^*(c_\psi) + (\lambda_2)^*(c_1(N))$$

where $c_1(N)$ stands for the first Chern class of (N,η).

Proof. By definition we have $c_{\psi\times \mathrm{id}_N} = c_1(T^{\mathrm{V}}P_{\psi\times \mathrm{id}_N})$. Then it follows by Prop. 2.6 that

$$\begin{aligned}
c_{\psi\times \mathrm{id}_N} &= c_1(T^{\mathrm{V}}P_{\psi\times \mathrm{id}_N}) \\
&= c_1((\lambda_1)^*(T^{\mathrm{V}}P_\psi) \oplus (\lambda_2)^*(TN)) \\
&= c_1((\lambda_1)^*(T^{\mathrm{V}}P_\psi)) + c_1((\lambda_2)^*(TN)) \\
&= (\lambda_1)^*(c_\psi) + (\lambda_2)^*(c_1(N)).
\end{aligned}$$

□

A similar result holds for the coupling forms of the fibrations P_ψ and $P_{\psi\times \mathrm{id}_N}$.

Lemma 2.8. *If $\tilde{\omega}$ is a coupling form of the Hamiltonian fibration $\pi : P_\psi \longrightarrow S^2$, then $(\lambda_1)^*(\tilde{\omega}) + (\lambda_2)^*(\eta)$ is a coupling form of the fibration $\pi_0 : P_\psi \times N \longrightarrow S^2$.*

The proof follows from Lemma 2.5 and the definition of the coupling form. We will write $\tilde{\omega} \oplus \eta$ for the coupling form $(\lambda_1)^*(\tilde{\omega}) + (\lambda_2)^*(\eta)$ on $P_\psi \times N \simeq P_{\psi\times \mathrm{id}_N}$.

Remark. In the proof of the main theorem it will be important to note the following. Let $A \in H_2(P_{\psi\times \mathrm{id}_N};\mathbb{Z})$ be any class such that $(\lambda_2)_*(A) = 0$. Then it follows from Lemmas 2.7 and 2.8 that

$$c_{\psi\times \mathrm{id}_N}(A) = c_\psi((\lambda_1)_*(A)), \tag{4}$$

and

$$\tilde{\omega} \oplus \eta(A) = \tilde{\omega}((\lambda_1)_*(A)). \tag{5}$$

3. Small quantum homology and Seidel's homomorphism

In this section we will review the concepts needed to define Seidel's representation. We will follow closely the exposition and notations of D. McDuff and D. Salamon [4].

Let ψ be a loop in the group of Hamiltonian diffeomorphisms of (M,ω) and $\pi : P_\psi \longrightarrow S^2$ the Hamiltonian fibration associated to the loop ψ. Consider $\tilde{\omega}$ a coupling form on the fibration. Then for a large positive constant K the form $\Omega := \tilde{\omega} + K\pi^*(\omega_0)$ on P_ψ is a nondegenerate 2-form, where ω_0 is an area form on S^2. Is important to note that Ω and $\tilde{\omega}$ induced the same horizontal distribution on P_ψ. Denote by $\mathcal{J}(P_\psi,\pi,\Omega)$ the set of almost complex structures J on P_ψ that are Ω-compatible and such that the projection map $\pi : (P_\psi, J) \longrightarrow (S^2, j_0)$ is holomorphic. Here j_0 is an arbitrary complex structure on S^2. Recall that Ω-compatible means in particular that $\Omega(Ju,Jv) = \Omega(u,v)$; and $\pi : (P_\psi, J) \longrightarrow (S^2, j_0)$ is holomorphic if $j_0 \circ (d\pi) = (d\pi) \circ J$. Since for any $J \in \mathcal{J}(P_\psi,\pi,\Omega)$ the projection map π is holomorphic, then J preserves the vertical tangent space of P_ψ. Also since J is a Ω-compatible almost complex structure, J preserves the horizontal distribution of P_ψ.

Consider a spherical class $A \in H_2(P_\psi;\mathbb{Z})$, that is A is in the image of the Hurewicz homomorphism $\pi_2(P_\psi) \longrightarrow H_2(P_\psi)$. Then if $J \in \mathcal{J}(P_\psi,\pi,\Omega)$, let $\mathcal{M}(A;J)$ be the moduli space of J-holomorphic sections of P_ψ that represent the class A,

$$\mathcal{M}(A;J) = \{u : S^2 \longrightarrow P_\psi \mid \bar{\partial}_J(u) = 0, \pi \circ u = \mathrm{id}_{S^2}, [u] = A\}.$$

Here $\bar{\partial}_J$ stands for the Cauchy-Riemann equation

$$\bar{\partial}_J(u) = \frac{1}{2}\left(du + J \circ du \circ j_0\right).$$

where j_0 is a fix complex structure on S^2.

To assure that the moduli space $\mathcal{M}(A;J)$ is a smooth finite dimensional manifold, one considers the linearized operator

$$D_u : \Omega^0(S^2, u^*(TP_\psi)) \longrightarrow \Omega^{0,1}(S^2, u^*(TP_\psi))$$

of $\bar{\partial}_J$. One finds that there is a subset $\mathcal{J}_{reg}(P_\psi, \pi, \Omega) \subset \mathcal{J}(P_\psi, \pi, \Omega)$ such that if J is in $\mathcal{J}_{reg}(P_\psi, \pi, \Omega)$, then $\mathcal{M}(A; J)$ is a smooth manifold of dimension $2n + 2c_\psi(A)$, where the symplectic manifold (M, ω) has dimension $2n$. Moreover $\mathcal{M}(A; J)$ carries a natural orientation. The set $\mathcal{J}_{reg}(P_\psi, \pi, \Omega)$ is characterized by the fact that J is regular if and only if for every holomorphic J-curve $u \in \mathcal{M}(A; J)$ the operator D_u is surjective. For the details see ([4], Ch. 3).

Denote by $\mathcal{M}_k(A; J)$ be the moduli space of J-holomorphic sections with k marked points. That is

$$\mathcal{M}_k(A; J) = \{(u, z_1, \ldots, z_k) \mid u \in \mathcal{M}(A; J), z_i \in S^2, z_i \neq z_j, \forall i \neq j\}.$$

This moduli space has dimension $\mu := 2n + 2c_\psi(A) + 2k$. Consider the evaluation map $ev : \mathcal{M}_k(A; J) \longrightarrow (P_\psi)^k$ given by $ev(u, z_1, \ldots, z_k) = (u(z_1), \ldots, u(z_k))$. Now one would like the map ev to represent a cycle in the homology of $(P_\psi)^k$. Actually the map ev is a pseudocycle if the manifold P_ψ is monotone. In the case at hand, Hamiltonian fibrations, is enough to impose this condition on the fiber M rather than on the whole manifold P_ψ. A symplectic manifold (M, ω) is said to be **monotone** if there is $\lambda > 0$ such that

$$\omega(A) = \lambda c_1(A)$$

for all $A \in \pi_2(M)$. Then if (M, ω) is monotone, if follows that ev is a pseudocycle of dimension μ in $(P_\psi)^k$. That is, it defines a homology class in $H_*((P_\psi)^k)$ of degree μ.

With this at hand we can define the corresponding Gromov-Witten invariants of P_ψ. However in order to define Seidel's representation one studies the moduli space of sections with one *fixed* marked point. Fix a point z_0 in the base S^2 and let $\iota : M \longrightarrow P_\psi$ be the inclusion of the fiber above the base point z_0. Then the moduli space

$$\mathcal{M}_1^{\mathbf{w}}(A; J) = \{(u, z_0) \mid u \in \mathcal{M}(A; J)\}.$$

is a smooth oriented manifold of dimension $2n + 2c_\psi(A)$, where $\mathbf{w} = \{z_0\}$ stands for the fixed marked point. Moreover the evaluation map $ev_{\mathbf{w}} : \mathcal{M}_1^{\mathbf{w}}(A; J) \longrightarrow P_\psi$ is a pseudocycle in P_ψ. If we consider the inclusion map $\iota : M \longrightarrow P_\psi$, then $\iota^{-1} \circ ev_{\mathbf{w}}$ represents a pseudocycle in M of degree $2n + 2c_\psi(A)$.

Let $H_*(M)$ be the torsion-free part of the group $H_*(M; \mathbb{Z})$. Then the Gromov-Witten invariant is defined as the homomorphism $\mathrm{GW}_{A,1}^{P_\psi, \mathbf{w}} : H_{-2c_\psi(A)}(M) \longrightarrow \mathbb{Z}$ given by

$$\mathrm{GW}_{A,1}^{P_\psi, \mathbf{w}}(\alpha) = (\iota^{-1} \circ ev_{\mathbf{w}}) \cdot_M \alpha$$

where $\cdot_M$ denotes the cycle intersection product in $H_*(M)$ and $A \in H_2(P_\psi; \mathbb{Z})$ a spherical class. Geometrically, $\mathrm{GW}_{A,1}^{P_\psi, \mathbf{w}}(\alpha)$ is the number of holomorphic sections of $\pi : P_\psi \longrightarrow S^2$ such that $u(z_0)$ lies in the cycle X, where $\alpha = [X] \in H_*(M)$.

Consider the universal Novikov ring Λ^{univ} defined as

$$\Lambda^{\text{univ}} = \left\{ \sum_{s \in \mathbb{R}} r_s t^s \mid r_s \in \mathbb{Z}, \#\{s > c : r_s \neq 0\} < \infty \text{ for every } c \in \mathbb{R} \right\}$$

and the graded polynomial ring $\Lambda := \Lambda^{\text{univ}}[q, q^{-1}]$ where q has degree 2. Then the small quantum homology of (M, ω) with coefficients in Λ is defined as

$$QH_*(M; \Lambda) := H_*(M) \otimes_{\mathbb{Z}} \Lambda.$$

Actually, $QH_*(M;\Lambda)$ is a ring under quantum product. The ring structure of $QH_*(M;\Lambda)$ will be describe below.

Then Seidel's homomorphism $\mathcal{S} : \pi_1(\mathrm{Ham}(M,\omega)) \longrightarrow QH_*(M;\Lambda)$ is defined as

$$\mathcal{S}(\psi) = \sum_{A \in H_2^{sec}(P_\psi)} \mathcal{S}_A(\psi) \otimes q^{-c_\psi(A)} t^{-\tilde{\omega}_\psi(A)} \tag{6}$$

where the sum runs over all spherical classes A that can be realized by a section. That is, a section $u : S^2 \longrightarrow P_\psi$ such that $[u] = A$. And $\mathcal{S}_A(\psi)$ is the homology class in $H_{2n+2c_\psi(A)}(M)$ determined by the relation

$$\mathrm{GW}_{A,1}^{P_\psi,\mathbf{w}}(\alpha) = \mathcal{S}_A(\psi) \cdot_M \alpha$$

for all $\alpha \in H_*(M)$. Observe that $\mathcal{S}(\psi)$ has degree $2n$ in $QH_*(M;\Lambda)$.

As pointed out in the introduction, at a first glimpse formula (6) of Seidel's representation looks different from that of McDuff-Tolman [5]. When (M,ω) admits a Hamiltonian circle action ψ, then the definition of $\mathcal{S}(\psi)$ simplifies. For instance P_ψ is isomorphic with the Borel quotient $S^3 \times_{S^1} M$, where S^3 corresponds to the total space of the Hopf fibration $S^3 \longrightarrow S^2$. Hence if $p \in M$, is a fixed point then the inclusion of p into M, induces a section $\sigma_{\max} : S^2 \longrightarrow S^3 \times_{S^1} M \simeq P_\psi$. Thus there is a preferred section $\sigma_{\max}$ when (M,ω) admits a Hamiltonian circle action. Further the index of the summation in Eq. (6), that is $A \in H_2^{sec}(P_\psi)$ can be substituted by $\sigma_{\max} + B$ where $B \in H_2(M,\mathbb{Z})$ is a spherical class. The rest of the details can be found in Prop. 3.3 of [5].

4. Quantum product and the Künneth formula

So far we have only described the additive structure of quantum homology $QH_*(M;\Lambda) = H_*(M) \otimes_{\mathbb{Z}} \Lambda$. However $QH_*(M;\Lambda)$ has the structure of a ring where the operation is called quantum product.

The quantum product is defined in terms of Gromov-Witten invariants, which are a slide different from the ones discussed in the previous section. Consider (M,ω) a closed monotone symplectic manifold, homogeneous elements $a,b,c \in H_*(M)$, and $A \in H_2(M)$ a spherical class. Then we have the Gromov-Witten invariant $\mathrm{GW}_{A,3}^M(a,b,c)$, which is the number of holomorphic curves that represent the class A and intersect the cycles that represent the classes a, b and c. Here the degree of the homology classes must satisfy the equation $\deg(a)+\deg(b)+\deg(c) = 4n - 2c_1(A)$, otherwise the invariant is defined as zero. (See [4], Ch. 7.) Now let $\{e_\nu\}_{\nu\in I}$ be a base of the free $\mathbb{Z}$-module $H_*(M)$, and $\{e_\nu^*\}_{\nu\in I}$ be the dual basis with respect to the intersection product. That is $e_\nu^* \cdot e_\mu = \delta_{\nu,\mu}$. Then if $a,b \in H_*(M)$ are homogeneous classes the quantum product $a * b$ is defined as

$$a * b = \sum_{\nu\in I} \sum_A \mathrm{GW}_{A,3}^M(a,b,e_\nu) e_\nu^* \otimes q^{-c_1(A)} t^{-\omega(A)},$$

where c_1 is the first Chern class of (M,ω), and the sum runs over all spherical classes $A \in H_2(M)$. Observe that $\deg(a*b) = \deg(a)+\deg(b)-2n$. Finally the quantum product extends Λ-linearly to all $QH_*(M;\Lambda)$. Note that the identity element under quantum multiplication corresponds to the fundamental class $1 = [M] \in H_{2n}(M)$.

An important fact about quantum homology is that the Künneth formula holds under a mild constraint. Let (M,ω) and (N,η) be closed symplectic manifolds which are monotone with the *same constant*. Thus the Gromov-Witten invariants of $M \times N$ are well defined. Let $a,b,c \in H_*(M \times N)$ be homogeneous classes such that the projections to $H_*(M)$ are denoted by a_1, b_1 and c_1, and similarly $a_2, b_2, c_2 \in H_*(N)$. Then we have the following relation between the Gromov-Witten invariants of M, N and $M \times N$,

$$\mathrm{GW}^{M\times N}_{A,3}(a,b,c) = \mathrm{GW}^{M}_{A_1,3}(a_1,b_1,c_1)\mathrm{GW}^{N}_{A_2,3}(a_2,b_2,c_2) \tag{7}$$

where $A \in H_2(M \times N)$ is a spherical class and A_1, A_2 correspond to the projection of A to $H_2(M)$ and $H_2(N)$ respectively. As a consequence we get the Künneth formula for quantum homology

$$QH_*(M \times N;\Lambda) \simeq QH_*(M;\Lambda) \otimes_\Lambda QH_*(N;\Lambda).$$

For more details see [4]. With this at hand we conclude that the maps κ and κ' that appear in the main theorems are ring homomorphisms under quantum multiplication.

Lemma 4.9. *Let (M,ω) and (N,η) be closed symplectic manifolds which are monotone with the same constant. Then the map of Thm. 1.1,*

$$\kappa : QH_*(M;\Lambda) \longrightarrow QH_{*+\dim(N)}(M \times N;\Lambda)$$

is a ring homomorphism under the quantum product.

Proof. Let $\alpha_1, \alpha_2 \in H_*(M)$. We must show that

$$(\alpha_1 \otimes [N]) * (\alpha_2 \otimes [N]) = (\alpha_1 * \alpha_2) \otimes [N].$$

This is a consequence of the Künneth formula for the quantum homology ring. For

$$\begin{aligned}(\alpha_1 \otimes [N]) * (\alpha_2 \otimes [N]) &= (\alpha_1 * \alpha_2) \otimes ([N] * [N]) \\ &= (\alpha_1 * \alpha_2) \otimes [N]\end{aligned}$$

since the fundamental class $[N]$ is the identity in $QH_*(N;\Lambda)$ under quantum multiplication. Thus κ is a ring homomorphism. □

A similar argument shows that the map κ' is a ring homomorphism.

Lemma 4.10. *Let (M,ω) be a closed symplectic manifold such that $\pi_2(M)$ is trivial. Then the map of Thm. 1.3,*

$$\kappa' : QH_*(M;\Lambda) \longrightarrow QH_{*+*}(M \times M;\Lambda)$$

is a ring homomorphism under the quantum product.

Proof. By the Künneth formula, write the map κ' as

$$\kappa' : QH_*(M;\Lambda) \longrightarrow QH_*(M;\Lambda) \otimes_\Lambda QH_*(M;\Lambda)$$

where $\kappa'(x) = x \otimes x$. If follows again from the Künneth formula that for $x, y \in QH_*(M;\Lambda)$,

$$\begin{aligned}\kappa'(x * y) &= (x * y) \otimes (x * y) \\ &= (x \otimes x) * (y \otimes y) \\ &= \kappa'(x) * \kappa'(y).\end{aligned}$$

Therefore κ' is a ring homomorphism under quantum multiplication. □

5. Proof of the main result

Let (M,ω) and (N,η) be closed symplectic manifolds as in Thm. 1.1. That is M is monotone and $\pi_2(N)=0$. Then the product symplectic manifold $(M\times N,\omega\oplus\eta)$ is also monotone, and therefore Seidel's representation is well defined on $(M\times N,\omega\oplus\eta)$.

Lemma 5.11. *Let $J\in\mathcal{J}(P_\psi,\pi,\Omega)$ where $\tilde\omega$ is a coupling form of $\pi:P_\psi\longrightarrow S^2$ and $\Omega=\tilde\omega+K\pi^*(\omega_0)$ as in Section 3. If J' is a η-compatible almost complex structure on TN, then $J\oplus J'\in\mathcal{J}(P_\psi\times N,\pi_0,\tilde\omega\oplus\eta+K\pi_0^*(\omega_0))$.*

Proof. Let $p:P_\psi\times N\longrightarrow P_\psi$ be the projection map. Then $\pi\circ p=\pi_0$. On $P_\psi\times N$ we have the symplectic form $\Omega'=\tilde\omega\oplus\eta+K\pi_0^*(\omega_0)$. Since $p^*\circ\pi^*=\pi_0^*$ then $\Omega'=\Omega\oplus\eta$ on $T(P_\psi\times N)\simeq TP_\psi\oplus TN$. Thus if J is an almost complex structure on TP_ψ which is Ω-compatible and J' a η-compatible almost complex structure on TN, we have

$$\begin{aligned}\Omega'(J\oplus J',J\oplus J') &= \Omega(J,J)\oplus\eta(J',J')\\ &= \Omega\oplus\eta\\ &= \Omega'.\end{aligned}$$

Hence $J\oplus J'$ is an Ω'-compatible almost complex structure on $T(P_\psi\times N)$.

Assume that J is such that the projection $\pi:(P_\psi,J)\longrightarrow(S^2,j_0)$ is holomorphic. That is $d\pi\circ J=j_0\circ d\pi$. Since $d\pi_0=d\pi\circ dp$, we have

$$\begin{aligned}d\pi_0\circ(J\oplus J') &= d\pi\circ dp\circ(J\oplus J')\\ &= (d\pi\circ J)\oplus 0\\ &= (j_0\circ d\pi)\oplus 0\\ &= j_0\circ d\pi_0.\end{aligned}$$

Therefore, $J\oplus J'\in\mathcal{J}(P_\psi\times N,\pi_0,\tilde\omega\oplus\eta+K\pi_0^*(\omega_0))$. □

The next proposition, is basically a restatement of Eq. (7), but for the Gromov-Witten invariants that are involved in the definition of the Seidel representation.

Proposition 5.12. *Let $A\in H_2(P_{\psi\times\mathrm{id}_N};\mathbb{Z})$ be a spherical class. Denote by $A_1:=(\lambda_1)_*(A)$ the induced spherical class in $H_2(P_\psi;\mathbb{Z})$. Then*

$$\mathrm{GW}_{A,1}^{P_{\psi\times\mathrm{id}_N},\mathbf{w}}(\alpha\otimes[\mathrm{pt}])=\mathrm{GW}_{A_1,1}^{P_\psi,\mathbf{w}}(\alpha)$$

for all $\alpha\in H_*(M)$.

Proof. Let $\tilde\omega$ be a coupling form of $\pi:P_\psi\longrightarrow S^2$ and $J\in\mathcal{J}_{reg}(P_\psi,\pi,\Omega)$. Thus if J' is a η-compatible almost complex structure on TN, it follows from Lemma 5.11, that $J\oplus J'\in\mathcal{J}(P_\psi\times N,\pi_0,\tilde\omega\oplus\eta+K\pi_0^*(\omega_0))$. We must show that $J\oplus J'$ is regular.

Let $u:S^2\longrightarrow P_\psi\times N$ be a $(J\oplus J')$-holomorphic section that represents the class $A\in H_2(P_\psi\times N;\mathbb{Z})$. Since $\pi_2(N)=0$, we may assume that $u=(u_0,q_0)$, where $u_0:S^2\longrightarrow P_\psi$ is a J-holomorphic section. Since J is regular and u_0 is J-holomorphic, we know that the linearized operator

$$D_{u_0}:\Omega^0(S^2,(u_0)^*(TP_\psi))\longrightarrow\Omega^{0,1}(S^2,(u_0)^*(TP_\psi))$$

is onto. For the curve $u=(u_0,q_0)$, we have the linearized operator

$$D_u:\Omega^0(S^2,(u_0)^*(TP_\psi)\oplus(S^2\times\mathbb{R}^{2m}))\longrightarrow\Omega^{0,1}(S^2,(u_0)^*(TP_\psi)\oplus(S^2\times\mathbb{R}^{2m}))$$

where $\dim(N) = 2m$. In this situation the operator D_u splits as the sum of D_{u_0} and $\bar{\partial}$. See [4], Rmk. 6.7.5. But the Cauchy-Riemann operator $\bar{\partial}$ is also surjective, thus it follows that D_u is also onto. Therefore $J \oplus J'$ is regular.

Henceforth, $\mathcal{M}_1^{\mathbf{w}}(A; J \oplus J')$ and $\mathcal{M}_1^{\mathbf{w}}(A_1; J)$ are smooth oriented manifolds and can be use to compute the corresponding Gromov-Witten invariant. Let $\alpha \in H_*(M)$, since $\pi_2(N) = 0$ we have the same intersection points for the pseudocycle

$$\iota^{-1} \circ ev_{\mathbf{w}} : \mathcal{M}_1^{\mathbf{w}}(A_1; J) \longrightarrow M$$

with α; and the pseudocycle

$$\iota_0^{-1} \circ ev_{\mathbf{w}} : \mathcal{M}_1^{\mathbf{w}}(A; J \oplus J') \longrightarrow M \times N.$$

with $\alpha \otimes [\mathrm{pt}]$. Hence $\mathrm{GW}_{A,1}^{P_{\psi\times \mathrm{id}_N}, \mathbf{w}}(\alpha \otimes [\mathrm{pt}]) = \mathrm{GW}_{A_1,1}^{P_\psi, \mathbf{w}}(\alpha)$. □

Now by Prop. 5.12, there is a similar relation between the homology classes $\mathcal{S}_A(\psi)$ and $\mathcal{S}_A(\psi \times \mathrm{id}_N)$.

Lemma 5.13. *Let $A \in H_2(P_{\psi\times \mathrm{id}_N}; \mathbb{Z})$ and $A_1 \in H_2(P_\psi; \mathbb{Z})$ as in Prop. 5.12. Then the identity*

$$\mathcal{S}_A(\psi \times \mathrm{id}_N) = \mathcal{S}_{A_1}(\psi) \otimes [N]$$

holds in $H_*(M \times N)$.

Proof. Let $\alpha \in H_*(M)$ and $\beta \in H_*(N)$ such that the sum of the degrees of α and β is $-2c_\psi(A_1)$. Then by the definition of the invariant $\mathrm{GW}_{A_1,1}^{P_\psi, \mathbf{w}}$,

$$\begin{aligned} (\mathcal{S}_{A_1}(\psi) \otimes [N]) \cdot_{M\times N} (\alpha \otimes \beta) &= (\mathcal{S}_{A_1}(\psi) \cdot_M \alpha) \cdot ([N] \cdot_N \beta) \\ &= \mathrm{GW}_{A_1,1}^{P_\psi, \mathbf{w}}(\alpha) \cdot ([N] \cdot_N \beta). \end{aligned} \tag{8}$$

The terms in this equation are all equal to zero unless α has degree $-2c_\psi(A_1)$ and β has degree 0, that is $\beta = [\mathrm{pt}]$. In this case, note that $[N] \cdot_N [\mathrm{pt}] = 1$.

On the other hand by the definition of $\mathrm{GW}_{A,1}^{P_{\psi\times \mathrm{id}_N}, \mathbf{w}}$ we have

$$\mathcal{S}_A(\psi \times \mathrm{id}_N) \cdot_{M\times N} (\alpha \otimes \beta) = \mathrm{GW}_{A,1}^{P_{\psi\times \mathrm{id}_N}, \mathbf{w}}(\alpha \otimes \beta). \tag{9}$$

Since the class α is in $H_*(M)$, then by definition of the Gromov-Witten invariant, it follows that $\mathrm{GW}_{A,1}^{P_{\psi\times \mathrm{id}_N}, \mathbf{w}}(\alpha \otimes \beta)$ is zero unless $\beta = [\mathrm{pt}]$. Hence from Prop. 5.12, Eqs. (8) and (9) are equal. That is,

$$(\mathcal{S}_{A_1}(\psi) \otimes [N]) \cdot_{M\times N} (\alpha \otimes \beta) = \mathcal{S}_A(\psi \times \mathrm{id}_N) \cdot_{M\times N} (\alpha \otimes \beta)$$

for all $\alpha \in H_*(M)$ and $\beta \in H_*(N)$. Therefore $\mathcal{S}_{A_1}(\psi) \otimes [N] = \mathcal{S}_A(\psi \times \mathrm{id}_N)$. □

Proof of Thm. 1.1. First of all, since $\pi_2(N)$ is trivial we have that

$$(\lambda_1)_* : H_2(P_{\psi\times \mathrm{id}_N}; \mathbb{Z}) \longrightarrow H_2(P_\psi; \mathbb{Z})$$

induces a one-to-one correspondence between the section classes of P_ψ and the section classes of $P_{\psi\times \mathrm{id}_N}$. That is, $H_2^{sec}(P_\psi) \simeq H_2^{sec}(P_{\psi\times \mathrm{id}_N})$. Hence the sum on the definition of the elements $\mathcal{S}(\psi)$ and $\mathcal{S}(\psi \times \mathrm{id}_N)$ is defined over the same set.

For $A \in H_2^{sec}(P_{\psi\times \mathrm{id}_N}; \mathbb{Z})$, we have that $(\lambda_2)_*(A) = 0$ since $\pi_2(N) = 0$. Therefore

$$c_{\psi\times \mathrm{id}_N}(A) = c_\psi(A_1) \quad \text{and} \quad \tilde{\omega}_{\psi\times \mathrm{id}_N}(A) = \tilde{\omega}_\psi(A_1)$$

by Eqs. (4) and (5). Finally from Lemma 5.13 we have $\mathcal{S}_A(\psi \times \mathrm{id}_N) = \mathcal{S}_{A_1}(\psi) \otimes [N]$. Therefore $\mathcal{S}(\psi \times \mathrm{id}_N) = \mathcal{S}(\psi) \otimes [N]$. □

Consider the group homomorphism $\tau_0 : \pi_1(\mathrm{Ham}(M,\omega)) \longrightarrow \pi_1(\mathrm{Ham}(M\times M, \omega\oplus\omega))$ defined as $\tau_0(\psi) = \mathrm{id}_M \times \psi$. Define the map

$$\kappa_0 : QH_*(M;\Lambda) \longrightarrow QH_{*+\dim(M)}(M\times M;\Lambda)$$

on homogeneous elements by $\kappa_0(\alpha\otimes q^r t^s) = (\alpha\otimes[M])q^r t^s$, where $\alpha\in H_*(M)$, and extend it Λ-linearly to all $QH_*(M;\Lambda)$. Then as in Thm 1.1, we have that

$$\mathcal{S}\circ\tau_0 = \kappa_0\circ\mathcal{S}. \tag{10}$$

Then Thm. 1.1 together with Eq. (10) provide a a proof of Thm. 1.3,

Proof of Thm. 1.3. Observe that $\tau'(\psi) = \tau(\psi)\circ\tau_0(\psi)$. Then applying Seidel's representation we get from Thm. 1.1 and Eq. (10) that

$$\begin{aligned}\mathcal{S}\circ\tau'(\psi) &= \mathcal{S}(\tau(\psi)\circ\tau_0(\psi))\\ &= \mathcal{S}(\tau(\psi))*\mathcal{S}(\tau_0(\psi))\\ &= (\kappa\circ\mathcal{S}(\psi))*(\kappa_0\circ\mathcal{S}(\psi))\\ &= (\mathcal{S}(\psi)\otimes[M])*([M]\otimes\mathcal{S}(\psi)).\end{aligned}$$

Then by the Künneth formula and the fact that $[M]$ is the indentity on $QH_*(M;\Lambda)$ we get

$$\begin{aligned}\mathcal{S}\circ\tau'(\psi) &= (\mathcal{S}(\psi)\otimes[M])*([M]\otimes\mathcal{S}(\psi))\\ &= (\mathcal{S}(\psi)*[M])\otimes([M]*\mathcal{S}(\psi))\\ &= \mathcal{S}(\psi)\otimes\mathcal{S}(\psi)\\ &= \kappa'(\mathcal{S}(\psi)).\end{aligned}$$

□

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