A Quick View of Lagrangian Floer Homology

Andrés Pedroza

Abstract In this note we present a brief introduction to Lagrangian Floer homology and its relation with the solution of Arnol'd conjecture, on the minimal number of non-degenerate fixed points of a Hamiltonian diffeomorphism. We start with the basic definition of critical point on smooth manifolds, in oder to sketch some aspects of Morse theory. Introduction to the basics concepts of symplectic geometry are also included, with the idea of understanding the statement of Arnol'd Conjecture and how is related to the intersection of Lagrangian submanifolds.

1 Introduction

Many elegant results in mathematics have to deal with the fixed-point-set of a function. For example: Brouwer fixed-point theorem, Lefschetz fixed-point theorem, Banach fixed-point theorem and Poincaré-Birkhoff theorem, just to name a few. Furthermore, these results are fundamental in their own area of mathematics and have interesting consequences in diverse areas of mathematics; differential equations, topology and game theory among others. Symplectic geometry has its own fixed-point theorem, which was conjectured by V. Arnol'd [1] in 1965. The Arnol'd Conjecture was motivated by Poincaré-Birkhoff theorem: An area-preserving diffeomorphism of the annulus which maps the boundary circles to themselves in different direction, must have at least two fixed points.

The generalization of Poincaré-Birkhoff theorem fits in symplectic geometry and not in volume-preserving geometry. The Arnol'd Conjecture establishes a lower bound on the number of fixed points a Hamiltonian diffeomorphism in terms of the topology of the manifold. The fixed points of a Hamiltonian diffeomorphism, (in fact any diffeomorphisms) can be seen as the intersection of its graph and the di-

Andrés Pedroza
Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo No. 340, Colima, Col., Mexico 28045 e-mail: andres_pedroza@ucol.mx

agonal. In the context of symplectic geometry, is the intersection of two Lagrangian submanifolds.

In 1987, A. Floer [10] developed a homological theory that focused on the intersection of Lagrangian submanifolds. In particular, under some hypotheses, he proved the Arnol'd Conjecture for a particular class of closed symplectic manifolds. This theory is called Lagrangian Floer homology.

In these notes we sketch how Lagrangian Floer homology is defined. In fact we review some aspects of Morse theory from its basics; like non-degenerate critical points, the Hessian, flow lines of the gradient vector field up to Morse homology. The reason being, that Lagrangian Floer homology emulates in many aspects Morse homology. Also we cover the basics of symplectic manifolds and Hamiltonian diffeomorphisms. The last section deals with Lagrangian Floer homology and how it is used to prove the Arnol'd Conjecture.

For the basic notions of differential geometry the reader can look at [41]; for the aspects of symplectic geometry [7] and [23]; and also [24] where the analytical aspect of holomorphic curves is covered. For details and proofs on the construction of Lagrangian Floer homology see [3], [32] and [33]. For an excellent introduction to Fukaya categories see [4]; and [40] for a detail treatment of the subject.

2 Morse-Smale Functions

Let M be a smooth manifold of dimension n and $f : M \to \mathbb{R}$ a smooth function. A point $p \in M$ is called a *critical point* of f if the differential $df_p : T_pM \to \mathbb{R}$ at p is the zero map. Denote by $\mathrm{Crit}(f)$ the set of critical points of f. Notice that $\mathrm{Crit}(f)$ can be the empty set, however if M is compact then it is not empty, since a smooth function on M has a maximum and a minimum.

Let $p \in M$ be a critical point of f and $(x_1, \ldots, x_n)$ a coordinate chart about p. The *Hessian matrix* of f at p relative to the chart $(x_1, \ldots, x_n)$, is the $n \times n$ matrix

$$\mathrm{Hess}(f, p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right).$$

A critical point p is said to be *non-degenerate* if the matrix $\mathrm{Hess}(f, p)$ is non-singular. Note that the Hessian matrix is symmetric, hence if it is non-singular its eigenvalues are real and non-zero. The *index* of f at a non-degenerate critical point p, which is denoted by $\mathrm{ind}(f; p)$, is defined as the number of negative eigenvalues of the Hessian matrix at p.

The definition of the index at a non-degenerate critical point given above depends on the coordinate system; however it can be shown that the is independent of the coordinate system about the the critical point.There is an alternative definition of the index of a function at a non-degenerate critical point, that does not needs a coordinate system. For a critical point $p \in M$ of f define the bilinear form

$$df_p^2 : T_pM \times T_pM \to \mathbb{R}$$

as $df_p^2(X,Y) := X(\widetilde{Y}f)$, where $\widetilde{Y}$ is any vector field on M whose value at p is Y. Notice that since p is a critical point of f, the bilinear form df_p^2 is symmetric,

$$0 = df_p[\widetilde{X},\widetilde{Y}] = [\widetilde{X},\widetilde{Y}]_p(f) = \widetilde{X}_p(\widetilde{Y}f) - \widetilde{Y}_p(\widetilde{X}f).$$

In this context, p is called *non-degenerate* if the bilinear symmetric form df_p^2 is non-degenerate. The *index* of f at p is defined as the number of negative eigenvalues of the symmetric bilinear form df_p^2. The two definitions given of non-degenerate critical point agree. The same applies for the two definitions of the index of a non-degenerate critical point. For further details, see [3, Ch. 1] and [29].

Definition 1. A smooth function $f : M \to \mathbb{R}$ for which all of its critical points are non-degenerate is called a *Morse function.*

Now we consider some examples in the case when $M = \mathbb{R}^2$. The origin is the only critical point of the function $f(x,y) = x^2 + y^2$. Moreover is a non-degenerate critical point and its index is zero. The origin is also the only non-degenerate critical point of the functions $g(x,y) = x^2 - y^2$ and $h(x,y) = -x^2 - y^2$. In these cases the index at the origin is 1 and 2 respectively. These three examples describe the general behavior of a function on $\mathbb{R}^2$ near the origin when it is a non-degenerate critical point. The precise statement on the behavior of a function near a non-degenerate critical point is given by Morse lemma.

Theorem 1 (Morse Lemma). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that the origin is a non-degenerate critical point of index λ. Then there exists a coordinate chart $(u_1,\ldots,u_n)$ about the origin such that*

$$f(u_1,\ldots,u_n) = f(0) - u_1^2 - \cdots - u_\lambda^2 + u_{\lambda+1}^2 + \cdots + u_n^2.$$

It goes without saying that Morse lemma also holds for smooth functions defined on arbitrary manifolds. A consequence of Morse lemma, as stated above, is that there exists a neighborhood about the origin in $\mathbb{R}^n$ so that it is the only critical point in such neighborhood.

Corollary 1. *Non-degenerate critical points of a smooth function are isolated.*

Note that a Morse function defined on a compact manifold has finitely many critical points.

The main reason behind the study of Morse functions is to understand the topology of the manifold. Thus for a smooth function $f : M \to \mathbb{R}$ and $a \in \mathbb{R}$ define the *level set*

$$M_a := \{x \in M | f(x) \leq a\} \subset M.$$

Notice that when a_0 is the absolute minimum of f, then M_a is empty for every $a < a_0$. And in the case when a_1 is the absolute maximum of f, then $M_a = M$ for every $a_1 \leq a$.

Now we explain what we mean by understanding the topology of the manifold; one aspect is that the manifold can be constructed from information from a fixed Morse function on it. Consider a compact manifold M, a smooth Morse function $f : M \to \mathbb{R}$ and for simplicity assume that $p_0,\ldots,p_k$ are all the critical points, with $\lambda_i = \mathrm{ind}(f;p_i)$ and $\lambda_i < \lambda_{i+1}$ for $i \in \{0,\ldots,k-1\}$. Thus f achieves its minimum at p_0 and $\mathrm{ind}(f;p_0) = 0$; and it achieves its maximum at p_k and $\mathrm{ind}(f;p_k) = n$. In order to build the manifold M from the critical points of f, one starts with the point $M_{f(p_0)} = \{p_0\}$. Then from Theorem 2 below, it follows that M_a has the same homotopy type has $M_{f(p_0)}$ for $a \in (f(p_0), f(p_1))$. By an *λ-cell* we mean a space homeomorphic to the closed ball of dimension λ. Hence, M_a is homeomorphic to the n-cell for $a \in (f(p_0), f(p_1))$.

The next step is to analyze the next non-degenerte critical point $p_1 \in M$. In this case for $a \in (f(p_1), f(p_2))$, it follows that M_a has the same homotopy type as $M_{f(p_0)}$ with a λ_1-cell attached. That is $M_a \simeq M_{f(p_0)} \cup_g e_{\lambda_1}$, where $g : \partial(e_{\lambda_1}) \to M_{f(p_0)}$ is a gluing function. This process continues at every critical point. That is for $a \in (f(p_i), f(p_{i+1}))$ the space M_a as the same homotopy type has to $M_{f(p_{i-1})}$ with an attached λ_i-cell. The last step asserts that $M_a \simeq M_{f(p_{k-2})} \cup e_{\lambda_{k-1}}$ for $a \in (f(p_{k-1}), f(p_k))$; that is M_a is homeomorphic to M minus an open ball. Therefore M is homeomorphic to M_a with a n-ball attached. Note that the change of topology between the level sets occurs precisely at the critical points of f. Below, we carry out the same process described above for $\mathbb{R}P^1$.

Therefore when $f : M \to \mathbb{R}$ is a Morse function, is possible to describe the topology of the level sets M_a as a increases; in particular the topology of M. Furthermore, there is an alternative approach to understand the topology of M using a Morse function. This is called Morse homology and it will be describe in Section 3.

Theorem 2. *Let $f : M \to \mathbb{R}$ be a Morse function.*

- *If f has no critical value in $[a,b]$, then M_a is diffeomorphic to M_b.*
- *If f has only one critical value in $[a,b]$ of index λ, then M_b has the same homotopy type as that of $M_a \cup_g e_\lambda$, for some gluing function g.*

As above, $M_a \cup e_\lambda$ means that e_λ is attached to M_a by some gluing function $g : \partial(e_\lambda) \to M_a$. Note that $\partial(e_\lambda)$ is diffeomorphic to $S^{\lambda-1}$. In the next example, we show how Theorem 2 is used to obtain the whole manifold M, by attaching one λ-cell at a time.

Example 1. Consider the real projective space $\mathbb{R}P^n$, the set of lines through the origin in $\mathbb{R}^{n+1}$. A point in $\mathbb{R}P^n$ is represented in homogeneous coordinates as $[x_0 : \ldots : x_n]$. Let $a_0,\ldots,a_n$ be distinct real numbers, define $f : \mathbb{R}P^n \to \mathbb{R}$ by

$$f([x_0 : \cdots : x_n]) = \frac{a_0x_0^2 + \cdots + a_nx_n^2}{x_0^2 + \cdots + x_n^2}.$$

So defined f is smooth and since the $a_j{}'s$ are distinct it has $n+1$ non-degenerate critical points, that are $p_0 := [1 : 0 : \cdots : 0], p_1 := [0 : 1 : \cdots : 0], \ldots, p_n := [0 : \cdots 0 : 1]$. Thus f is a Morse function; moreover the critical point p_j has index j.

The reader is encouraged to verify the statements made above. And also to get the same conclusions for the case of the complex projective space $\mathbb{C}P^n$ with the function

$$f([z_0:\cdots:z_n]) = \frac{a_0|z_0|^2+\cdots+a_n|z_n|^2}{|z_0|^2+\cdots+|z_n|^2}.$$

Now we look at the particular case of $\mathbb{R}P^1$; recall that $\mathbb{R}P^1$ is diffeomorphic to the circle. In this particular case take $a_0=0$ and $a_1=1$, so f takes the form

$$f([x_0:x_1]) = \frac{x_1^2}{x_0^2+x_1^2}.$$

In this case f has only one critical point of index 0, namely at $[1:0]$. It also has only one critical point of index 1, at $[0:1]$. These points correspond to the maximum and minimum of f. In terms of Theorem 2 the circle is obtained as follows. We start with the 0-cell that is just a point, that is $M_0=\{[1:0]\}$. Since f has no critical values in the interval $[0,1/2]$ other than 0, then from Theorem 2 if follows that M_0 has the same homotopy type has $M_{1/2}$. Notice that $M_{1/2}$ is a semicircle, the south hemisphere. Next comes the other critical point $[0:1]$. It has index 1; thus a 1-cell is attached to $M_{1/2}$. That is, the two points of $\partial(e_1)$ get glued to $M_{1/2}$ to obtain the circle. See Figure 1.

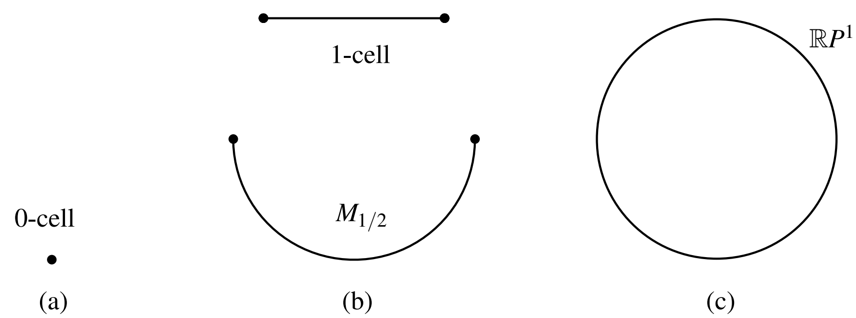

Fig. 1 Morse decomposition of $\mathbb{R}P^1$ with respecto to f. In (a) the 0-cell that corresponds to p_0 of index 0. In (b) the submanifold $M_{1/2}$, diffeomorphic to a point, is attached a 1-cell that corresponds to the point p_1 of index 1. Finally, (c) the result after attaching the 1-cell.

An important aspect to consider is the existence of Morse functions on a given manifold. It turns out that there are plenty of Morse functions. More precisely, the set of Morse functions on a closed manifold is C^2-dense in the space of smooth functions. The reason that the C^2-topology is needed is because the concept of non-degenerate critical points involves derivatives up to second-order. In theory, is not to difficult to understand the topology of M via a Morse function as above. Next we take this idea a step further to recover the homology of M.

Fix a Riemannian metric g on M and let $\langle\cdot,\cdot\rangle$ be the induced inner product on its tangent bundle. The *gradient vector field*, $\mathrm{grad}(f)$, of the function $f: M\to\mathbb{R}$ is

defined by the equation
$$\langle \mathrm{grad}(f), X\rangle = X(f)$$
for every vector field X on M. Notice that if p is a critical point of f, then $\mathrm{grad}(f)_p = 0$. And conversely, if $\mathrm{grad}(f)_p = 0$ then p is a critical point of f. Therefore $\mathrm{Crit}(f)$ equals the zero set of $\mathrm{grad}(f)$.

In order to simplify the exposition, from now on we assume that M is compact. Denote by $\theta : \mathbb{R}\times M \to M$ the flow of the **negative** gradient vector field of f. Thus for $x \in M$
$$\left.\frac{\partial\theta(t,x)}{\partial t}\right|_{t=0} = -\mathrm{grad}(f)_x.$$
The reason to consider the negative gradient vector field is only a matter of convention. Note that $-\mathrm{grad}(f) = \mathrm{grad}(-f)$ and if p is a non-degenerate critical point of f, then $\mathrm{ind}(f;p) = n - \mathrm{ind}(-f;p)$ where n is the dimension of M. Also notice that $-\mathrm{grad}(f)(f) < 0$ outside the set of critical points of f, hence $-\mathrm{grad}(f)$ points in the direction in which f is decreasing. The way to think about the index of a non-degenerate critical point is the number of linearly independent directions in which the $-\mathrm{grad}(f)$ decreases. Let p be a point where $-\mathrm{grad}(f)$ vanishes, then consider all points of M that under the flow θ converge to p as t goes to infinity;
$$W^s(f,p) := \left\{ x\in M \,\middle|\, \lim_{t\to+\infty}\theta(t,x) = p \right\}.$$
Similarly,
$$W^u(f,p) := \left\{ x\in M \,\middle|\, \lim_{t\to-\infty}\theta(t,x) = p \right\},$$
the set of all points in M that have p has a source. Since $-\mathrm{grad}(f)$ vanishes at p, then the critical point p is a fixed under the flow, hence $p\in W^s(f,p)$ and $p \in W^u(f,p)$. The submanifolds $W^u(f,p)$ and $W^s(f,p)$ are called the *unstable manifold* and *stable submanifold* of f at p, respectively.

Theorem 3. *If p is a non-degenerate critical point of f, then $W^u(f;p)$ is a smooth submanifold of M of dimension* $\mathrm{ind}(f;p)$.

Instead, if we consider the function $-f$ the set critical non-degenerate points of f and $-f$ agree. Moreover $W^u(f;p) = W^s(-f;p)$ and $W^s(f;p) = W^u(-f;p)$. Hence $W^s(f;p)$ is also a smooth submanifold of M of dimension $n - \mathrm{ind}(f;p)$.

Example 2. Let $M = S^2$ be the unit sphere in $\mathbb{R}^3$ centered at the origin and $f : S^2 \to \mathbb{R}$ defined as $f(x,y,z) = z$. Then the poles $N = (0,0,1)$ and $S = (0,0,-1)$ are the critical points of f. Furthermore they are non-degenerate, N has index 2 and S has index 0.

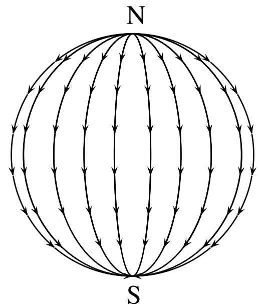

Fig. 2 The flow lines of the gradient vector field of $f(x,y,z) = z$ on S^2 with respect to the standard Riemannian structure.

Consider the Riemannian structure on S^2 induced from the standard Riemannian structure on $\mathbb{R}^3$. Then $-\text{grad}(f)$ is the vector field that points downwards, and

$$W^u(f,N) = S^2 \setminus \{S\}, W^s(f,N) = \{N\}, W^u(f,S) = \{S\} \text{ and } W^s(f,S) = S^2 \setminus \{N\}.$$

Example 3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \cos(2\pi x) + \cos(2\pi y).$$

So defined f induces a smooth function on the flat two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, which we still denote by f. There are 4 non-degenerate critical points on the torus, $p_1 = [0,0], p_2 = [0,1/2], p_3 = [1/2,0]$ and $p_4 = [1/2,1/2]$, of index $2,1,1$ and 0 respectively. Consider the Riemannian structure on $\mathbb{T}^2$ induced from the canonical Riemannian structure on $\mathbb{R}^2$. Then the flow of $-\text{grad}(f)$ can be seen in Figure 3.

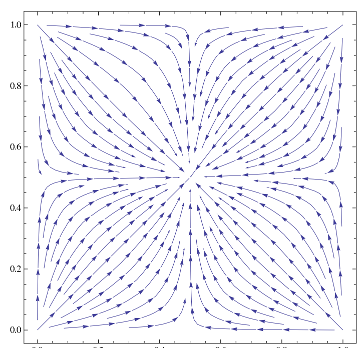

Fig. 3 The flow lines of the gradient vector field of $\cos(2\pi x) + \cos(2\pi y)$ on $\mathbb{T}^2$ with respect to the flat Riemannian structure.

Notice that there are only two lines that connect p_2 to p_4. And a 1-dimensional family of flow lines that connect p_1 to p_4, whose points determine four open connected components of the torus.

Also Figure 3 gives a description of the stable and unstable submanifolds. Observe that every interior point of $[0,1]\times[0,1]$, lies in a flow line that ends at p_4. That is,

$$W^s(f;p_4)=\mathbb{T}^2\setminus\{\partial([0,1]\times[0,1])\}.$$

Similarly we have that $W^u(f;p_1)$ equals

$$\mathbb{T}^2\setminus\{p_2,p_3,p_4\}\cup\{(x,1/2)|x\in[0,1]\setminus\{1/2\}\}\cup\{(1/2,y)|y\in[0,1]\setminus\{1/2\}\},$$

and

$$\begin{aligned} W^u(f;p_2) &= \{(x,1/2)|x\in[0,1]\setminus\{1/2\}\},\\ W^u(f;p_3) &= \{(1/2,y)|y\in[0,1]\setminus\{1/2\}\}. \end{aligned}$$

Let p and q be non-degenerate critical points of a smooth function $f:M\to\mathbb{R}$. Then the set $W^u(f;p)\cap W^s(f;q)$ consists of points of M that belong to a flow line $u:\mathbb{R}\to M$ of $-\mathrm{grad}(f)_{u(t)}$ that connects p to q; that is

$$\frac{du}{dt}(t)=-\mathrm{grad}(f)_{u(t)},\quad \lim_{t\to-\infty}u(t)=p \quad\text{and}\quad \lim_{t\to+\infty}u(t)=q. \tag{1}$$

We know from Theorem 3 that $W^u(f;p)$ and $W^s(f;q)$ are submanifolds of M, but their intersection might not be a smooth manifold. Hence a smooth function $f:M\to\mathbb{R}$ is said to satisfy the *Smale condition* if for any pair of critical points p and q, $W^u(f;p)$ and $W^s(f;q)$ intersect transversally. In particular $W^u(f;p)\cap W^s(f;q)$ is a submanifold of M.

The function that appears in Example 2 satisfies the Smale condition. In this example the intersection of any pair of stable and unstable submanifolds is either empty, a point, the sphere minus a point or the sphere minus two points. Also the Moorse function in Example 3 satisfies the Smale condition. In particular, notice that $W^u(f;p_2)\cap W^s(f;p_4)$ consists of two disjoint open intervals.

The type of functions that are of interest in this note are the Morse-Smale functions. For an arbitrary compact manifold and Riemannian metric, there always exists a Morse-Smale function. Furthermore, in some sense there are plenty of such functions. Then if (M,g) is a Riemannian manifold and $f:M\to\mathbb{R}$ a Morse-Smale function we write $\mathscr{M}(f;p,q)$ for the set of points of M that belong to a flow trajectory of $-\mathrm{grad}(f)$ that goes from p to q as in Eq. (1). Notice that in this case $\mathscr{M}(f;p,q)$ is a smooth submanifold of M of dimension $\mathrm{ind}(f;p)-\mathrm{ind}(f;q)$. Note that the submanifold $\mathscr{M}(f;p,q)$ admits a natural action of $\mathbb{R}$ defined as $(s.u)(t):=u(t+s)$ for $s\in\mathbb{R}$. The action is in fact free and the orbit space of this action is denoted by $\hat{\mathscr{M}}(f;p,q)$. Hence $\hat{\mathscr{M}}(f;p,q)$ is identified as the space of trajectories that joint p to q.

So defined, the space of points that belong to a flow line of $-\mathrm{grad}(f)$ that connect p to q, $\mathscr{M}(f;p,q)$, is not necessarily compact. For example, in the case of the two-spere in Example 2 we have that $\mathscr{M}(f;N,S)$ is $S^2\setminus\{N,S\}$. In this example if we add the critical points we obtain a compact space, namely the whole manifold S^2. Note that in this example $\hat{\mathscr{M}}(f;N,S)$ is diffeomorphic to S^1. But it is not always the case that by adding the critical points p and q to $\mathscr{M}(f;p,q)$ that it becomes a compact space. For instance, in the torus case of Example 3 the space $\mathscr{M}(f;p_1,p_4)\cup\{p_1,p_4\}$ is not compact.

In general, the way to compactify the space of trajectories $\hat{\mathscr{M}}(f;p,q)$ is by adding broken trajectories. A *broken trajectory* from p to q is a collection of flow lines $\{u_1,\dots,u_r\}$ of $-\mathrm{grad}(f)$ such that u_j connects the critical points x_j to x_{j+1} for $j\in\{1,\dots,r\}$ where $p=x_1$ and $q=x_{r+1}$. Consider the bigger set of flow lines that connect p to q, namely usal flow trajectories plus broken trajectories,

$$\overline{\mathscr{M}}(f;p,q) := \hat{\mathscr{M}}(f;p,q)\cup\{\text{broken trajectories from } p \text{ to } q\}.$$

Recall that the index of critical points of f decreases along flow lines. Hence the number of flow lines that form a broken flow lines is less than $\mathrm{ind}(f;p)-\mathrm{ind}(f;q)$. Hence if $\mathrm{ind}(f;p)-\mathrm{ind}(f;q)=1$ there are no broken trajectories connecting p to q and $\overline{\mathscr{M}}(f;p,q)=\hat{\mathscr{M}}(f;p,q)$. That is, $\hat{\mathscr{M}}(f;p,q)$ is compact in this case and it consists of finitely many points.

The proof of the next result can consulted in [3, Chp. 3] and [38].

Proposition 1. *Let (M,g) be a closed Riemannian manifold, $f:M\to\mathbb{R}$ a Morse-Smale function and p,q critical points of f. Then the natural action of $\mathbb{R}$ on $\mathscr{M}(f;p,q)$ is free. Moreover $\overline{\mathscr{M}}(f;p,q)$ is smooth and compact of dimension* $\mathrm{ind}(f;p)-\mathrm{ind}(f;q)-1$.

The important case that would be relevant later on is the case when $\mathrm{ind}(f;p)-\mathrm{ind}(f;q)=2$. Usually in this case the space $\hat{\mathscr{M}}(f;p,q)$ is not compact, so we must add broken trajectories. Hence $\overline{\mathscr{M}}(f;p,q)$ is a finite collection of closed intervals and circles.

In Example 3 consider $u_1(t):=[0,t]$ and $u_2(t):=[t,1/2]$, for $t\in(0,1/2)$, two flow lines of $-\mathrm{grad}(f)$. The flow line u_1 connects p_1 to p_2, and u_2 connects p_2 to p_4. Hence $\{u_1,u_2\}$ is a broken trajectory that connects p_1 to p_4. Note that $\hat{\mathscr{M}}(f;p_1,p_4)$ is diffeomorphic to four copies of $(0,1)$; and there are eight broken trajectories that must be added to obtain $\overline{\mathscr{M}}(f;p,q)$. For instance, $\{u_1,u_2\}$ is one of them. Henceforth $\overline{\mathscr{M}}(f;p,q)$ is diffeomorphic to four copies of $[0,1]$.

3 Morse Homology

We are going to define the Morse-Witten complex of $(M;f,g)$; the Riemannian manifold and the Morse-Smale function. For simplicity we will use $\mathbb{Z}_2$ coefficients, keep in mind that it is possible to use integer coefficients. In order to define Morse homology with integer coefficients, one must prove that is possible to have a coherent system of orientation on the compact moduli spaces. In the case of $\mathbb{Z}_2$ coefficients the orientation of the moduli spaces is irrelevant, only the boundary components of the moduli spaces of dimension two are important. See for example [38], where they use integer coefficients. Also we drop the dependence of the Riemannian metric from the notation. Denote by $\mathrm{Crit}_\lambda(f)$ the set of critical points of index λ and by $C_\lambda(f)$ the $\mathbb{Z}_2$-vector space generated by the elements of $\mathrm{Crit}_\lambda(f)$. For $\lambda \notin \{0,1,\ldots n\}$, define $C_\lambda(f)$ to be the trivial vector space. If p and q are critical points of f such that $\mathrm{ind}(f;p) = \mathrm{ind}(f;q)+1$, then by Proposition 1 $\hat{\mathscr{M}}(f;p,q)$ is a finite set of points. Denote by $\#_{\mathbb{Z}_2}\hat{\mathscr{M}}(f;p,q)$ the number of points of $\hat{\mathscr{M}}(f;p,q)$ module 2.

The boundary operator, $\partial_\lambda : C_\lambda(f) \to C_{\lambda-1}(f)$, is the linear map defined on generators $p \in C_\lambda(f)$ as

$$\partial_\lambda(p) := \sum_{q\in \mathrm{Crit}_{\lambda-1}(f)} \#_{\mathbb{Z}_2}\hat{\mathscr{M}}(f;p,q)\, q.$$

Notice that if $\hat{\mathscr{M}}(f;p,q)$ is zero-dimensional, then $\mathscr{M}(f;p,q)$ consists of finitely many lines that connect p to q. This geometric description of $\mathscr{M}(f;p,q)$ is useful when computing the boundary operator ∂; this will be seen for instance below in Example 4. The reason why ∂_λ is called the boundary operator is given by the next result.

In order to compute $\partial_{\lambda-1}\circ\partial_\lambda$ one must consider the moduli spaces $\overline{\mathscr{M}}(f;p,r)$ where $\mathrm{ind}(f;p) - \mathrm{ind}(f;r) = 2$. For $p \in \mathrm{Crit}_{\lambda-2}(f)$,

$$\partial_{\lambda-1}\partial_\lambda(p) := \sum_{r\in \mathrm{Crit}_{\lambda-2}(f)}\ \sum_{q\in \mathrm{Crit}_{\lambda-1}(f)} \#_{\mathbb{Z}_2}(\#\hat{\mathscr{M}}(f;p,q)\cdot \#\hat{\mathscr{M}}(f;q,r))\, r$$

where $\#\hat{\mathscr{M}}(f;p,q)$ stands for the number of points of $\hat{\mathscr{M}}(f;p,q)$. Notice that $\overline{\mathscr{M}}(f;p,r)$ is a one-dimensional compact manifold; hence it is the union of a finite collection of closed intervals and circles. Hence its boundary consists of a even number of points which are

$$\cup_{q\in \mathrm{Crit}_{\lambda-1}(f)} \#\hat{\mathscr{M}}(f;p,q)\cdot \#\hat{\mathscr{M}}(f;q,r)$$

and correspond to the broken trajectories from p to r that go thru q.

Theorem 4. *The operator satisfies* $\partial_{\lambda-1}\circ\partial_\lambda = 0$.

The complex $(\oplus_\lambda C_\lambda(f),\partial)$ is called the *Morse-Witten complex* of $(M;f,g)$. Its homology

$$\mathrm{MH}_\lambda(M;f,g) := \frac{\mathrm{Ker}\,\partial_\lambda}{\mathrm{Im}\,\partial_{\lambda+1}}$$

is called the Morse homology of $(M;f,g)$ with $\mathbb{Z}_2$-coefficients. Note that the relevant moduli spaces $\overline{\mathscr{M}}(f;p,q)$ for the definition of Morse homology are those whose dimension is at most two.

Remark 1. As mention above is possible to define Morse homology with $\mathbb{Z}$ coefficients. For, $W^u(f,p)$ is an orientable submanifold of M for any critical point p. Hence, one fixes an orientation on $W^u(f,p)$ for every critical point. This yields an orientation on $W^u(f,p)\cap W^s(f,q)$ and hence on $\mathscr{M}(f;p,q)$ and $\hat{\mathscr{M}}(f;p,q)$. Thus if $\mathrm{ind}(f;p) = \mathrm{ind}(f;q)+1$, then $\overline{\mathscr{M}}(f;p,q) = \hat{\mathscr{M}}(f;p,q)$ is a finite set of points each of which has a sign. Set $n(f;p,q)$ to be the sum of these signs; then the boundary operator over $\mathbb{Z}$-coefficients is defined as

$$\partial_\lambda(p) := \sum_{q\in \mathrm{Crit}_{\lambda-1}(f)} n(f;p,q)\,q.$$

However the statement $\partial_{\lambda-1}\circ\partial_\lambda = 0$ is delicate in this case. One must take into consideration that the orientation of the moduli spaces of dimension two induced the right orientation on its boundary; the one-dimensional moduli spaces. For example see [38].

Recall from Example 2, that on S^2 we defined a Morse function with the poles N and S as critical points of index 2 and 0 respectively. The Riemannian structure on the sphere was induced from the canonical Riemannian structure on $\mathbb{R}^3$. Further, we calculated the stable and unstable submanifolds of N and S. From this calculation, it follows that f is a Morse-Smale function. Therefore $C_0(f) = \mathbb{Z}_2\langle S\rangle$, $C_2(f) = \mathbb{Z}_2\langle N\rangle$ and the boundary operator is the zero map. Hence

$$\mathrm{MH}_\lambda(S^2;f,g) = \begin{cases} \mathbb{Z}_2 & \text{if } \lambda = 0,2 \\ 0 & \text{if } \lambda \neq 0,2. \end{cases}$$

Example 4. In this example we consider the function on the two-dimensional torus $\mathbb{T}^2$ defined in Example 3. Notice that the function is Morse-Smale. Hence $C_0(f) = \mathbb{Z}_2\langle p_4\rangle, C_1(f) = \mathbb{Z}_2\langle p_2,p_3\rangle$, and $C_2(f) = \mathbb{Z}_2\langle p_1\rangle$. Counting trajectory flow lines, we get $\partial p_1 = 2p_2+2p_3 = 0$, $\partial p_2 = 0$, $\partial p_3 = 0$, and $\partial p_4 = 0$. Therefore,

$$\mathrm{MH}_\lambda(\mathbb{T}^2;f,g) = \begin{cases} \mathbb{Z}_2 & \text{if } \lambda = 0,2 \\ \mathbb{Z}_2\times\mathbb{Z}_2 & \text{if } \lambda = 1 \\ 0 & \text{if } \lambda \neq 0,1,2 \end{cases}$$

Summing up, we started with a smooth closed manifold M, then we choose a smooth function f and a Riemannian metric g, such that the critical points of f were non-degenerate and the intersection of the stable and unstable submanifolds were transversal. With all these data, we defined the Morse homology of $(M;f,g)$ At the end, Morse homology is a topological invariant of the manifold; that is, is independent of the function and the Riemannian metric. Furthermore it recovers the ordinary homology of the manifold. See [39] and [42].

Theorem 5. *Let M be a compact manifold, g a Riemannian metric and f a Morse-Smale function. Then $\mathrm{MH}_*(M;f,g)$ is independent of the function and the metric. Moreover $\mathrm{MH}_\lambda(M;f,g) \simeq H_\lambda(M;\mathbb{Z}_2)$ as vectors spaces for every λ.*

In the words of R. Bott [6], *Morse theory indomitable*. Here we barely treated the subject and its consequences. The reader is encouraged to learn more about the subject in [6], [21], [29] and in the beautiful monograph of J. Milnor, [28]. One application of Morse theory is the handlebody decomposition of a manifold; a much finer result than that stated in Theorem 5. In particular the Bott periodicity theorem is a marvelous consequence of Morse theory. Another typical consequence of Morse theory are the Morse inequalities. Here the problem is to determine lower bounds for the number of critical points of a fixed index of a Morse function. Denote by $b_\lambda(M)$ the λ-Betti number of M, that is the rank of $H_\lambda(M;\mathbb{Z}_2)$.

Theorem 6 (Morse's inequalities). *Let M be a a closed manifold and $f : M \to \mathbb{R}$ a Morse function. Then*

$$\#\,\mathrm{Crit}_\lambda(f) \geq b_\lambda(M)$$

for every $0 \leq \lambda \leq n$.

Thus Morse theory gives a lower bound for the minimal number of critical points that a Morse function can have on a manifold. Finally we mention that some features of ordinary homology, for example Poincaré duality and product operations, can be described in the Morse homology setting. See [13], [39] and [42].

4 Symplectic Manifolds and Lagrangian Submanifolds

A *symplectic form* on a manifold M is 2-form ω that is closed $d\omega = 0$ and non-degenerate. Here non-degenerate means that at every $p \in M$ and every nonzero vector $v \in T_pM$ there exists a vector $u \in T_pM$ such that $\omega_p(v,u)$ is nonzero. In this case (M,ω) is called a *symplectic manifold*. The symplectic form been non-degenerate, implies that the dimension of M must be even. Unless otherwise stated from now on we assume that the dimension of (M,ω) is $2n$.

The first and fundamental example of a symplectic manifold is $(\mathbb{R}^{2n},\omega_0)$. Here we take $(x_1,y_1,\ldots,x_n,y_n)$ as coordinates in $\mathbb{R}^{2n}$ and the symplectic form is defined as

$$\omega_0 := dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n.$$

In the 2-dimensional case, a symplectic form is the same as a volume form. Hence an oriented surface together with a volume form is an example of a symplectic manifold.

Example 5. Let S^2 be the unit sphere in $\mathbb{R}^3$ centered at the origin. Then for $p \in S^2$ and $u,v \in T_pS^2$ define

$$\omega_p(u,v) := \langle p, u \times v \rangle$$

where $\langle\cdot,\cdot\rangle$ is the inner product and $\times$ is the cross product in $\mathbb{R}^3$. So defined ω is a non-degenerate 2-form on the sphere. By dimension reasons, $d\omega = 0$; therefore ω is a symplectic form on the unit 2-sphere.

Example 6. Another important class of examples of symplectic manifolds are cotangent bundles T^*N of any smooth manifold N. Let $\pi : T^*N \to N$ be the projection map and $\pi_{*,(q,v^*)} : T_{(q,v^*)}T^*N \to T_qN$ its differential at (q,v^*). Define the 1-form λ_{can} on T^*N at (q,v^*) as

$$\lambda_{\mathrm{can},(q,v^*)} := v^* \circ \pi_{*,(q,v^*)}.$$

Then the canonical symplectic form on T^*N is defined as $\omega_{\mathrm{can}} := -d\lambda_{\mathrm{can}}$. This example is particularly important in Classical Mechanics. In fact the roots of symplectic geometry go back to Classical Mechanics. For example see [2].

In particular if $N = \mathbb{R}^n$ with coordinates $(x_1,\ldots,x_n)$ and the fibre $T_x^*\mathbb{R}^n$ with coordinates $(y_1,\ldots,y_n)$, then $T^*\mathbb{R}^n$ has coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_n)$. In this case the 1-form defined above takes the form

$$\lambda_{\mathrm{can}} = \sum_{j=1}^{n} y_j dx_j.$$

Moreover $\omega_{\mathrm{can}} = -d\lambda_{\mathrm{can}} = \omega_0$; hence on $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$ we get the symplectic form defined at the beginning of this section.

Another source of examples of symplectic manifolds are Kähler manifolds. In particular the complex projective space $(\mathbb{C}P^n, \omega_{FS})$ admits a symplectic form called the *Fubini-Study symplectic form*, which is induced from the Fubini-Study hermitian metric. That is, if $U_j = \{[z_0 : \cdots : z_n] | z_j \neq 0\} \subset \mathbb{C}P^n$ is a canonical open set, then on U_j the Fubini-Study symplectic form is defined as

$$\omega_{FS} = \frac{i}{2}\partial\overline{\partial}\left(\log \frac{z_0\bar{z}_0 + \cdots + z_n\bar{z}_n}{z_j\bar{z}_j}\right).$$

Furthermore, the symplectic area of the complex line $\mathbb{C}P^1 \subset (\mathbb{C}P^n, \omega_{FS})$ is π.

It is also possible to create new symplectic manifolds from old ones. Cartesian product of symplectic manifolds is such an example; since this example will be of importance later on we explain it in Example 7. Nonetheless there are more ways to create new symplectic manifolds, such as: symplectic reduction, fibrations and blow ups just to name a few.

Example 7. Let (M,ω) and (N,η) a symplectic manifolds and consider $M\times N$ with projection maps π_M and π_N. Then $(M\times N, \pi_M^*(\omega) + \pi_N^*(\eta))$ is a symplectic manifold. In particular for the same symplectic manifold and projections maps $\pi_j : M\times M \to M$ for $j = 1,2$, we get the symplectic manifold $(M\times M, \pi_1^*(\omega) + \pi_2^*(\omega))$. However it will be more important to consider a different symplectic form on $M\times M$, namely $\pi_1^*(\omega) - \pi_2^*(\omega)$. Below we will see why the minus sign is important in the second term.

As mentioned above, the standard euclidean symplectic space $(\mathbb{R}^{2n}, \omega_0)$ is the fundamental example of a symplectic manifold. The reason is that locally any symplectic manifold looks like $(\mathbb{R}^{2n}, \omega_0)$.

Theorem 7 (Darboux). *Let (M,ω) be a symplectic manifold and $p \in M$. Then there exists a coordinate chart (U,ψ) about p such that*

$$\omega = \psi^*(\omega_0)$$

on U.

An important consequence of the above result is that symplectic manifolds do not have local invariants. Thus the techniques and methods used in symplectic geometry are different from those in Riemannian geometry.

As mentioned at the Introduction, we aim to give a broad overview of Lagrangian Floer homology; which is defined for compact and exact symplectic manifolds. A symplectic manifold (M,ω) is called *exact* if there exists a 1-form λ such that $\omega = d\lambda$. Each case has its own hypothesis and restrictions. In this note we will cover only the compact case. Thus from now on the symplectic manifold (M,ω) will be assumed to be closed, that is compact with no boundary. However to illustrate some concepts, some examples will take place on arbitrary symplectic manifolds.

A *Lagrangian submanifold L* of a symplectic manifold (M,ω) is an embedded submanifold $j: L \to M$ of dimension n such that $j^*(\omega)$ is identically zero. For example, the unit circle S^1 centered at the origin in $(\mathbb{R}^2,\omega_0)$ is a Lagrangian submanifold. More generally, any embedding of S^1 into a two-dimensional symplectic manifold is Lagrangian, for dimensional reasons.

On the complex projective space $(\mathbb{C}P^n,\omega_{FS})$, the real projective space submanifold

$$\mathbb{R}P^n = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \,|\, z_0,\ldots,z_n \in \mathbb{R}\}$$

is Lagrangian. Another important Lagrangian submanifold of $(\mathbb{C}P^n,\omega_{FS})$ is the *Clifford torus* defined as

$$\{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \,|\, |z_0| = \cdots = |z_n|\}.$$

Notice that $\mathbb{R}P^n$ and the Clifford torus meet in 2^n points, namely $[\pm 1 : \cdots : \pm 1]$.

In the case of the symplectic manifold $(T^*N, \omega_{\mathrm{can}} = -d\lambda_{\mathrm{can}})$ of Example 6, the zero section and a fiber are examples of Lagrangian submanifolds. In particular the subspaces $\{(x_1,0,x_2,0,\ldots 0,x_n,0)\}$ and $\{(0,y_1,0,y_2,0,\ldots,y_{n-1},0,y_n)\}$ of $(\mathbb{R}^{2n},\omega_0)$ are Lagrangian submanifolds. There is also another significant class of Lagrangian submanifolds of $(T^*N,-d\lambda_{\mathrm{can}})$. Let σ be a 1-form on N and consider it as a section $\sigma : N \to T^*N$; that is $\sigma(p) = \sigma_p(\cdot)$. Hence σ embeds N into T^*N and following the definition of the canonical 1-form we get that

$$\sigma^*(\lambda_{\mathrm{can}}) = \sigma.$$

Hence the graph of a 1-form is a Lagrangian submanifold of $(T^*N,-d\lambda_{\mathrm{can}})$ if and only if the 1-form is closed.

In the case of the symplectic manifold $(M\times M, \pi_1^*(\omega) - \pi_2^*(\omega))$ of Example 7, the diagonal $\Delta = \{(x,x) | x \in M\}$ is a Lagrangian submanifold. Notice that the minus sign in the second term of the symplectic form in fundamental to guarantee that Δ is Lagrangian. Below in Example 9 we will exhibit more Lagrangian submanifolds of $(M\times M, \pi_1^*(\omega) - \pi_2^*(\omega))$ that are related to symplectic diffeomorphisms of (M,ω); the diagonal Δ is the graph of the identity diffeomorphism.

The relevance of the symplectic manifold $(T^*N,-d\lambda_{\mathrm{can}})$ of Example 6, is that it is a symplectic model of a neighborhood whenever $N \subset M$ is a Lagrangian submanigfold, regardless of the symplectic manifod (M,ω). That is, if N is a Lagrangian submanifold of (M,ω) then a tubular neighborhood of it can be identified, in a suplectic way, with a neighborhood of the zero section of $(T^*N,-d\lambda_{\text{can}})$. That is, there is an analog of Darboux's Theorem for Lagrangian submanifolds, where the standard symplectic euclidean space $(\mathbb{R}^{2n},\omega_0)$ is replaced by the standard symplectic cotangent bundle $(T^*N,-d\lambda_{\text{can}})$. Recall from above that in the case $N=\mathbb{R}^n$, we showed that $(T^*\mathbb{R}^n,-d\lambda_{\text{can}})$ agrees with $(\mathbb{R}^{2n},\omega_0)$.

Theorem 8 (Weinstein). *Let (M,ω) be a symplectic manifold and L a Lagrangian submanifold. Then there exists a neighborhood U of L and a neighborhood V of L_0, the zero section of $(T^*L,-d\lambda_{\text{can}})$, that are diffeomorphic by $\psi: U\to V$ such that*

$$\omega=\psi^*(-d\lambda_{\text{can}})$$

and $\psi(L)=L_0$.

Using the fact that locally any symplectic manifold is equal to $(\mathbb{R}^{2n},\omega_0)$, is possible to show that there are many Lagrangian submanifolds in any given symplectic manifold (M,ω). The circle S^1 is Lagrangian submanifold of $(\mathbb{R}^2,\omega_0)$. Moreover we can make the radius arbitrary small, say $\varepsilon>0$, and still $S^1(\varepsilon)$ is a Lagrangian submanifold. Taking n copies of this example, it follows that the n-dimensional ε-torus $S^1(\varepsilon)\times\cdots\times S^1(\varepsilon)$ is a Lagrangian submanifold of $(\mathbb{R}^2,\omega_0)\times\cdots\times(\mathbb{R}^2,\omega_0)=(\mathbb{R}^{2n},\omega_0)$. Thus for a given symplectic manifold (M,ω) and ε small enough, by Darboux's Theorem we have that the n-dimensional ε-torus is a Lagrangian submanifold of (M,ω).

In the particular case of $(\mathbb{R}^2,\omega_0)$, we have that the two-dimensional torus $S^1\times S^1$ is a Lagrangian submanifold. Furthermore, the torus is the only oriented surface that can be embedded as a Lagrangian submanifold in $(\mathbb{R}^2,\omega_0)$.

5 Symplectic and Hamiltonian Diffeomorphisms

There are two types of symmetries associated to a symplectic manifold. Recall that we assumed that that symplectic manifold is closed. In the non-compact case, one has to consider diffeomorphisms with compact support. A diffeomorphism $\phi:(M,\omega)\to(M,\omega)$ is said to be a *symplectic diffeomorphism* if $\phi^*(\omega)=\omega$. The set of symplectic diffeomorphisms of (M,ω) forms a group under composition and is denoted by $\text{Symp}(M,\omega)$. In fact the group of symplectic diffeomorphisms is an infinite dimensional space, its Lie algebra consists of vector fields X such that the 1-form $\omega(X,\cdot)$ is closed.

Among the group of symplectic diffeomorphisms we have the second type of symmetries, called Hamiltonian diffeomorphisms. A symplectic diffeomorphism ϕ is called *Hamiltonian diffeomorphism* if there exists a path of symplectic diffeomorphisms $\{\phi_t\}_{0\le t\le 1}$ and a smooth function $H:[0,1]\times M\to\mathbb{R}$, such that $\phi_0=1_M$, $\phi_1=\phi$, and if X_t is the time-dependent vector field induced by the equation

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t,$$

then $\omega(X_t, \cdot) = dH_t$. The set of Hamiltonian diffeomorphisms is a group under composition and is denoted by $\mathrm{Ham}(M, \omega)$. As in the symplectic case, $\mathrm{Ham}(M, \omega)$ is an infinite dimensional space and its Lie algebra consists of vector fields X such that the 1-form $\omega(X, \cdot)$ is exact.

A Hamiltonian diffeomorphisms ϕ is called *autonomous*, if there exists a path $\{\phi_t\}$, as in the definition of Hamiltonian diffeomorphism, such that X_t is independent of t. In other words autonomous Hamiltonian diffeomorphisms are the image of the exponential map of Hamiltonian vector fields. Alternatively, the group $\mathrm{Ham}(M, \omega)$ can be described as the group generated by autonomous Hamiltonian diffeomorphisms [5].

Not only is $\mathrm{Ham}(M, \omega)$ a subset $\mathrm{Symp}(M, \omega)$, as the definition suggest; the group of Hamiltonian diffeomorphisms is a normal subgroup of the group of symplectic diffeomorphisms. As we explain below in most cases is a proper subgroup. Among other properties of $\mathrm{Ham}(M, \omega)$ is that it is connected with respect to the C^∞-topology; $\mathrm{Symp}(M, \omega)$ does not have to be connected. Further if $\mathrm{Symp}_0(M, \omega)$ is the connected component of the group of symplectic diffeomorphisms that contains the identity map and $H^1(M, \mathbb{R}) = 0$, then

$$\mathrm{Ham}(M, \omega) = \mathrm{Symp}_0(M, \omega).$$

For example $\mathrm{Ham}(\mathbb{C}P^n, \omega_{\mathrm{FS}}) = \mathrm{Symp}_0(\mathbb{C}P^n, \omega_{\mathrm{FS}})$ for $n \geq 1$. However if $H^1(M, \mathbb{R}) \neq 0$, $\mathrm{Ham}(M, \omega)$ is properly contained in $\mathrm{Symp}_0(M, \omega)$. Below we will see an example where $\mathrm{Ham}(M, \omega)$ is a proper subgroup of $\mathrm{Symp}_0(M, \omega)$.

An important remark about a Hamiltonian diffeomorphism is that its fixed point set of is non-empty. Recall that we are assuming that (M, ω) is closed; in the non-compact case the assertion is false. For instance, on $(\mathbb{R}^{2n}, \omega_0)$ a translation map is a Hamiltonian diffeomorphism that is fixed-point free. As for the case of compact symplectic manifolds it is straightforward to justify that the fixed point set is non-empty in the case of autonomous Hamiltonians. For if ϕ is an autonomous Hamiltonian, then there exists $\{\phi_t\}$ such that

$$\frac{d}{dt}\phi_t = X \circ \phi_t, \qquad \text{and} \qquad \omega(X, \cdot) = dH. \tag{2}$$

Since the manifold is assumed to be compact, then the set of critical points of $H : M \to \mathbb{R}$ is non-empty. But ω is non-degenerate, hence by Eqs. (2) the set of critical points of H coincides with the zero set of X. If X vanishes at p it follows by Eqs. (2) that p is a fixed point of the flow $\{\phi_t\}$; in particular p is a fixed point of $\phi_1 = \phi$.

The fact that Hamiltonian diffeomorphisms on compact manifold always have fixed points is not shared by symplectic diffeomorphisms that are not Hamiltonian.

Example 8. Consider the flat two-dimensional torus $(\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2, \omega = dx \wedge dy)$. For a fix $\alpha \in (0, 1)$, the translation map

$$\phi^{\alpha}[x,y] := [x+\alpha, y]$$

preserves the area and hence is symplectic diffeomorphism. Notice that since $\alpha \neq 0,1$, the map ϕ^{α} has no fixed points. Hence ϕ^{α} is not a Hamiltonian diffeomorphism for any $\alpha \in (0,1)$. Moreover ϕ^{α} lies in the identity component of the group of symplectic diffeomorphism. Therefore $\mathrm{Ham}(\mathbb{T}^2,\omega)$ is a proper subgroup of $\mathrm{Symp}_0(\mathbb{T}^2,\omega)$.

As mentioned above, symplectic diffeomorphisms give rise to Lagrangian submanifolds. In the next example we show how this is done and highlight the importance of this example in the study of fixed points of Hamiltonian diffeomorphisms.

Example 9. Let $\phi : (M,\omega) \to (M,\omega)$ be a symplectic diffeomorphism, thus $\phi^*(\omega) = \omega$. Then the graph of ϕ is an embedded submanifold of dimension $2n$ in $M \times M$, that is $j : M \to M \times M$ is given by $j(x) = (x, \phi(x))$ and its image is the graph of ϕ,

$$\mathrm{graph}(\phi) := \{(x, \phi(x)) | x \in M\}.$$

Furthermore the graph of ϕ is a Lagrangian submanifold of $(M \times M, \pi_1^*(\omega) - \pi_2^*(\omega))$; for

$$j^*(\pi_1^*(\omega) - \pi_2^*(\omega)) = \omega - \phi^*(\omega) = 0.$$

The above computation shows the relevance of the minus sign that appears in the symplectic form of $(M \times M, \pi_1^*(\omega) - \pi_2^*(\omega))$ in Example 7. In the case when $\phi : M \to M$ is a Hamiltonian diffeomorphism, when know that the fixed point set is non empty; further this set is in one-to-one correspondence with the intersection points of $\mathrm{graph}(\phi)$ with the diagonal Δ. As pointed out above, there are symplectic diffeomorphisms ϕ, such that $\mathrm{graph}(\phi)$ and Δ have no points in common. For instance ϕ^{α} of Example 8.

Notice that for any Hamiltonian diffeomorphism $\phi : (M,\omega) \to (M,\omega)$ and Lagrangian submanifold $L \subset (M,\omega)$, $\phi(L)$ is again a Lagrangian submanifold. An important fact that will be useful in the context of Lagrangian Floer homology is the following. A Lagrangian submanifold $L \subset (M,\omega)$ is called *non-displaceable* if for every Hamiltonian diffeomorphisms $\phi : (M,\omega) \to (M,\omega)$, the Lagrangian submanifolds L and $\phi(L)$ have points in common. Otherwise, L is called *displaceable*. Hence we are considering the intersection of two particular Lagrangian submanifold, L and $\phi(L)$. This is part of the phenomenon that Lagrangian Floer homology attempts to answer, intersection or non-intersection of Lagrangian submanifolds.

Consider the two-dimensional sphere (S^2,ω) with any area form, let us try to understand the intersection of a particular pair of Lagrangian submanifolds. Consider the Lagrangian submanifold L to be any circle that lies entirely in a hemisphere. Thus there always exists a rotation $\phi : (S^2,\omega) \to (S^2,\omega)$, which is in fact a Hamiltonian diffeomorphism of the 2-sphere, such that L and $\phi(L)$ have no points in common. That is L is displaceable. Now consider the case when L is such that both components U and V of $S^2 \setminus L$ have equal area. Recall that any Hamiltonian diffeomorphism preserves area; hence for any Hamiltonian diffeomorphism ϕ, $\phi(U) = U$

or $\phi(U)\cap V$ is not empty. Then for any Hamiltonian diffeomorphism ϕ, we have that $L\cap\phi(L)$ is non-empty if the Lagrangian submanifold is such that the two components of $S^2\setminus L$ have equal area. Hence the non-displaceable Lagrangian are precisely the embedded circles that split the sphere in two pieces of equal area. In fact one of the current problems in symplectic geometry is to determine which Lagrangian submanifolds are non-displaceable or displaceable.

The higher dimensional analog of the above example, for the case when L splits the sphere in two parts of equal area, is the Lagrangian submanifold $\mathbb{R}P^n$ in $(\mathbb{C}P^n,\omega_{FS})$. One of the triumphs of Lagrangian Floer homology is the proof that $\mathbb{R}P^n$ is non-displaceable. This result was proved by Y.-G. Oh in [30]; where he defined Lagrangian Floer homology for monotone Lagrangian submanifolds.

Now we go back to the case of Lagrangian submanifolds induced by symplectic diffeomorphisms as in Example 9. Hence let $\phi:(M,\omega)\to(M,\omega)$ be a symplectic diffeomorphism and $\mathrm{graph}(\phi)\subset(M\times M,\pi_1^*(\omega)-\pi_2^*(\omega))$ which is a Lagrangian submanifold. Note that in this example the Lagrangian submanifold $\mathrm{graph}(\phi)$ it is actually the image of the Lagrangian Δ under the symplectic diffeomorphisms $1\times\phi$ of $(M\times M,\pi_1^*(\omega)-\pi_2^*(\omega))$. That is

$$\mathrm{graph}(\phi)=(1\times\phi)(\Delta).$$

In fact when $\phi:(M,\omega)\to(M,\omega)$ is a Hamiltonian diffeomorphism we know that $(1\times\phi)(\Delta)\cap\Delta$ is non empty. The intersection points are in one-to-one correspondence with the fixed points of ϕ. In this case $(1\times\phi)$ is also a Hamiltonian diffeomorphism of $(M\times M,\pi_1^*(\omega)-\pi_2^*(\omega))$. Lagrangian Floer homology gives a stronger result, it shows that Δ is non displaceable, that is $\Phi(\Delta)\cap\Delta\neq\emptyset$ for **any** Hamiltonian diffeomorphisms Φ of $(M\times M,\pi_1^*(\omega)-\pi_2^*(\omega))$, not necessarily those induced from Hamiltonians of (M,ω). Moreover it gives a lower bound on the cardinality of $\Phi(\Delta)\cap\Delta$ under some non degeneracy conditions of Φ. That is it solves the Arnol'd Conjecture.

The problem of estimating the number of fixed points of a Hamiltonian diffeomorphism, is a particular case of the wider problem of estimating the number of intersection points of two Lagrangian submanifolds. In broad terms, that is the objective of Lagrangian Floer homology.

As seen in the definition, Hamiltonian diffeomorphisms have a strong connection with smooth functions. A manifestation of this connection was the nice link between the fact that a smooth function on a closed manifold admits critical points; and the fact the on a closed symplectic manifold the fixed point set of a Hamiltonian diffeomorphism is non-empty.

In 1965, V. Arnol'd [1] conjectured an analog result of Theorem 6, but for the case of Hamiltonian diffeomorphisms on closed symplectic manifolds instead of Morse functions on arbitrary manifolds. See also [2, Appendix 9]. His motivation was the Poincaré-Birkhoff annulus theorem: An area preserving diffeomorphism of the annulus such that the boundary circles are turned in different directions must have at least two fixed points. A fixed point $p\in M$ of a Hamiltonian diffeomorphism ϕ is said to be *non-degenerate* if 1 is not an eigenvalue of the the linear map $\phi_{*,p}$:

$T_pM \to T_pM$. Note that non-degenerate fixed points are isolated, and in the case of a closed symplectic manifold there are a finite number of them.

Conjecture 1 (Arnol'd). Let (M,ω) be a closed symplectic manifold and ϕ a Hamiltonian diffeomorphism such that all of its fixed points are non-degenerate. Then

$$\#\{p \in M|\phi(p) = p\} \geq \sum_{j=0}^{2n} \text{Rank } H_j(M,\mathbb{R}).$$

For two-dimensional symplectic manifolds, the conjecture was proved by Y. Eliashberg [9]; in [8] C. C. Conley and E. Zehnder proved the conjecture for the symplectic torus manifold with the standard symplectic form; and for the complex projective space with the Fubini-Study symplectic form the conjecture was proved by B. Fortune and A. Weinstein in [12]. The real break through in solving Arnold's conjecture was made by A. Floer in [10].

In a series of papers A. Floer developed a homological theory based on holomorphic techniques, which were introduced by M. Gromov [18], and the new approach to Morse theory developed by E. Witten [42]. Under some assumption on the symplectic manifold A. Floer developed Hamiltonian Floer homology, using holomorphic cylinders, in order to find a lower bound to the number of fixed points of a Hamiltonian diffeomorphism. Then he generalized this approach to develop Lagrangian Floer homology, now using holomorphic stripes, in order to determine the minimum number of intersection points of a particular pair of Lagrangian submanifolds.

The Arnold's conjecture has been proved for arbitrary symplectic manifolds. Some reference for the proof of the conjecture, sometimes under some restrictions and others in full generality are: K. Fukaya and K. Ono [16], H. Hofer and D. Salamon [19], G. Liu and G. Tian [20], K. Ono [34], Y.-G. Oh [30], Y. Ruan [37].

Some of the techniques introduced in [16] on the proof of the Arnold's conjecture have been re-evaluated. For instance the Kuranishi structure on the moduli space of holomorphic strips $\mathscr{M}_J(p,q,L_0,L_1)$, that will be defined in the next section, as well as its virtual fundamental class. Recently J. Pardon [35] has given an alternative approach to this problem using techniques from homological algebra. There is also a series of articles by D. Mcduff and K. Wehrheim, [22] [26], [25] and [27], where they treat this problem using tools from analysis.

6 Lagrangian Floer Homology

The construction of Lagrangian Floer homology emulates to a certain extent the construction of Morse homology described above. The manifold in consideration to define it is a certain space of trajectories which is infinite dimensional; and the function defined on it is a certain action functional. The critical points turn out to be constant trajectories, that give rise to the differential complex used to define

Lagrangian Floer homology. Is important to point out while the construction of Lagrangian Floer homology follows the spirit of the construction of Morse homology, new complications emerge that were not present before. Just to have an idea of this, it suffices to say that Lagrangian Floer homology is not always defined do to the fact that the square of the differential map is not always equal to zero.

Let L_0 and L_1 be two compact Lagrangian submanifolds in (M,ω) that intersect transversally. Consider the space of smooth trajectories from L_0 to L_1,

$$\mathscr{P}(L_0,L_1) := \{\gamma : [0,1] \to M | \gamma \text{ is smooth, } \gamma(0) \in L_0 \text{ and } \gamma(1) \in L_1\}.$$

endowed with the C^∞-topology. Notice that the constant paths in $\mathscr{P}(L_0,L_1)$ are the ones that correspond to the intersection points $L_0 \cap L_1$. From now on we write $\mathscr{P}$ for $\mathscr{P}(L_0,L_1)$.

The space $\mathscr{P}$ is not necessarily connected, thus we fix $\hat{\gamma}$ in $\mathscr{P}$ and consider the component that contains $\hat{\gamma}$, which we denoted by $\mathscr{P}(\hat{\gamma})$. Relative to $\hat{\gamma}$ consider the universal covering space $\widetilde{\mathscr{P}}(\hat{\gamma})$ of $\mathscr{P}(\hat{\gamma})$. Elements of $\widetilde{\mathscr{P}}(\hat{\gamma})$ are denoted by $[\gamma,w]$ where w is a smooth path in $\mathscr{P}(\hat{\gamma})$ from $\hat{\gamma}$ to γ. That is $w : [0,1]\times[0,1] \to M$ is a smooth map such that $w(s,\cdot) \in \mathscr{P}(\hat{\gamma})$ for all $s \in [0,1]$, $w(0,\cdot) = \hat{\gamma}$ and $w(1,\cdot) = \gamma$.

The space $\widetilde{\mathscr{P}}(\hat{\gamma})$ is not the right space to define the action functional. The right space is the Novikov covering of $\mathscr{P}(\hat{\gamma})$, which is defined by an equivalence relation on $\widetilde{\mathscr{P}}(\hat{\gamma})$. For the sake of making the exposition less technical we are not going to define the Novikov covering of $\mathscr{P}(\hat{\gamma})$, instead we are going to impose strong assumptions on the symplectic manifold (M,ω) and the pair of Lagrangian submanifolds L_0 and L_1 in order to define the action functional on $\widetilde{\mathscr{P}}(\hat{\gamma})$ and carry out a similar procedure as in Morse theory. Thus from now on we assume that the symplectic manifold (M,ω) is such that

$$\int_{S^2} f^*\omega = 0$$

for every $[f] \in \pi_2(M)$, where f is a smooth representative. A symplectic manifold that satisfies this condition is said to be *symplectically aspherical*. The symplectic tori $(\mathbb{T}^{2n},\omega)$, for $n \geq 1$, are examples symplectically aspherical since $\pi_2(\mathbb{T}^{2n})$ is trivial. However there are plenty of symplectically aspherical manifolds with non trivial π_2, even in dimension four [17]. As we will see in the next paragraph, if (M,ω) is symplectically aspherical then the action functional is well defined on the covering $\widetilde{\mathscr{P}}(L_0,L_1;\hat{\gamma})$. This hypothesis on (M,ω) is also useful since it rules out the appearance of bubbles; that is holomorphic spheres attached to holomorphic strips. See for example []. As mentioned above, Lagrangian Floer homology is defined on more generally symplectic manifolds, even in the presence of bubbles.

Further we also assume that L_0 and L_1 are simply connected. Then the action functional $\mathscr{A} : \widetilde{\mathscr{P}}(L_0,L_1;\hat{\gamma}) \to \mathbb{R}$ is defined as

$$\mathscr{A}([\gamma,w]) = \int_{[0,1]\times[0,1]} w^*\omega,$$

that is, the symplectic area of $w([0,1]\times[0,1])\subset(M,\omega)$. To see that $\mathscr{A}$ is well defined, let (γ,w) and (γ,w') represent the same class. Then we have a map defined on the cylinder, $\overline{w}\#w' : S^1\times[0,1]\to M$ where $\overline{w}\#w'(s,0)$ is a loop in L_0 and $\overline{w}\#w'(s,1)$ is a loop in L_1. Since the Lagrangian submanifolds are assumed to be simply connected, there exists a 2-disk contained in L_0 whose boundary is the loop $\overline{w}\#w'(s,0)$. This observation also applies to L_1. That is, we added the caps to the cylinder to obtain topological a 2-sphere. Since (M,ω) is symplectically aspherical, the symplectic area of the 2-sphere is zero. Notice also that the symplectic area of each cap is zero, since the symplectic form is identically zero on Lagrangian submanifolds. Thus the symplectic area of the cylinder $\overline{w}\#w' : S^1\times[0,1]\to M$ is equal to zero. Then we have that

$$\begin{aligned} 0 &= \int_{S^1\times[0,1]} \overline{w}\#w^*\omega \;=\int_{[0,1]\times[0,1]} \overline{w}^*\omega + \int_{[0,1]\times[0,1]} w^*\omega \\ &= -\int_{[0,1]\times[0,1]} w^*\omega + \int_{[0,1]\times[0,1]} w^*\omega \end{aligned}$$

Hence it follows that the action functional is well defined on the covering $\widetilde{\mathscr{P}}(L_0,L_1;\hat{\gamma})$. In local coordinates the action functional takes the form

$$\mathscr{A}([\gamma,w]) = \int_0^1\int_0^1 \omega\left(\frac{\partial w}{\partial s},\frac{\partial w}{\partial t}\right)ds\,dt.$$

Lagrangian Floer homology is defined emulating the way Morse homology is defined. The finite dimensional manifold in Morse homology is replaced by the infinite dimensional space $\widetilde{\mathscr{P}}(L_0,L_1;\hat{\gamma})$. And the function into consideration is the action functional. However the analytical difficulties in this setting are more complex than in the Morse scenario.

To follow the path of Morse homology we need to define a Riemannian structure on $\widetilde{\mathscr{P}}(L_0,L_1;\hat{\gamma})$. Denote by $\mathscr{J}$ the space of almost complex structures on (M,ω). Recall that an almost complex structure $J\in\mathscr{J}$ on (M,ω) is said to be *ω-compatible* if for every nonzero vector v,

$$\omega(v,Jv)>0 \quad\text{ and }\quad \omega(J\cdot,J\cdot)=\omega(\cdot,\cdot).$$

For a ω-compatible almost complex structure J, we have that

$$g_J(\cdot,\cdot):=\omega(\cdot,J\cdot)$$

defines a Riemannian metric on (M,ω). Is important to note that compatible almost complex structures exist in abundance on any symplectic manifold. Let $J=\{J_t\}_{0\le t\le 1}$ be a smooth family of ω-compatible almost complex structures on (M,ω); hence we have $\{g_t\}_{0\le t\le 1}$ a smooth family of Riemannian metrics. Then on $\widetilde{\mathscr{P}}(\hat{\gamma})$ we define a Riemannian metric associated to $\{g_t\}_{0\le t\le 1}$ as

$$\langle\!\langle \xi_1,\xi_2\rangle\!\rangle := \int_0^1 g_t(\xi_1(t),\xi_2(t))dt$$

for ξ_1, ξ_2 in $T\widetilde{\mathscr{P}}(\hat{\gamma})$. As in the Morse theory case, we compute the gradient of $\mathscr{A}$: $\widetilde{\mathscr{P}}(\hat{\gamma}) \to \mathbb{R}$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$,

$$
\begin{aligned}
d\mathscr{A}_{([\gamma,w])}(\xi) &= \int_0^1 \omega\left(\frac{\partial \gamma}{\partial t}, \xi(t)\right) dt \\
&= \int_0^1 \omega\left(\frac{\partial \gamma}{\partial t}, J_t(-J_t\xi(t))\right) dt \\
&= \int_0^1 g_t\left(\frac{\partial \gamma}{\partial t}, -J_t\xi(t)\right) dt \\
&= \left\langle\!\!\left\langle \frac{\partial \gamma}{\partial t}, -J_t\xi \right\rangle\!\!\right\rangle \\
&= \left\langle\!\!\left\langle J_t\frac{\partial \gamma}{\partial t}, \xi \right\rangle\!\!\right\rangle.
\end{aligned}
$$

That is,

$$
\operatorname{grad}\mathscr{A}([\gamma, w]) = J_t\frac{\partial \gamma}{\partial t}.
$$

Since J_t is an automorphism of TM for each t, the gradient of $\mathscr{A}$ vanishes at $[\gamma, w]$ if and only if $\gamma : [0,1] \to (M, \omega)$ is a constant path. Thus the critical points of the action functional $\mathscr{A}$ are of the form $[\gamma, w]$ where γ is a constant path, corresponding to an intersection point of L_0 with L_1.

As in the Morse theory case, a flow line of $-\operatorname{grad}\mathscr{A}$ connecting p to q is a smooth function $u : \mathbb{R} \to \widetilde{\mathscr{P}}(\hat{\gamma})$ such that

$$
\frac{du}{ds} = -\operatorname{grad}\mathscr{A}, \quad \lim_{s\to-\infty} u(s) = p, \quad \text{and} \quad \lim_{s\to+\infty} u(s) = q. \tag{3}
$$

Unwrapping this, is better to write $u : \mathbb{R} \times [0,1] \to (M, \omega)$ and the first equation of Eqs. (3) as

$$
\frac{\partial u}{\partial s} + J_t\frac{\partial u}{\partial t} = 0. \tag{4}
$$

Note that Eq. (4) is the Cauchy-Riemann equation, $\overline{\partial}_J(u) = 0$, with respecto to the ω-compatible almost complex, $\{J_t\}$, structure on (M, ω). A smooth map $u : \mathbb{R} \times [0,1] \to (M, \omega)$ is said to be a *J-holomorphic strip* in (M, ω) if $\overline{\partial}_J(u) = 0$. Then the space of connecting flow lines (actually J-holomorphic strips in (M, ω, J)) that connect p with q is defined as as

$$
\begin{aligned}
\hat{\mathscr{M}}_J(p, q, L_0, L_1) := \{u : \mathbb{R} \times [0,1] \to (M, \omega) \mid\ & u \text{ is smooth, satisfies Eqs. (3)} \\
& \text{and } u(s, \cdot) \in \mathscr{P}(L_0, L_1)\}.
\end{aligned}
$$

In the case when (M, ω) is non compact but exact, and the Lagrangian submanifolds are still compact, one imposes an additional condition on the flow lines. That is to say, in addition to Eqs. (3) the map $u : \mathbb{R} \times [0,1] \to (M, \omega)$ is required to have finite energy,

$$\int_{\mathbb{R}\times[0,1]} u^*\omega < \infty. \tag{5}$$

In the case of a compact symplectic manifold, a holomorphic strip u with $u(s,\cdot) \in \mathscr{P}(L_0,L_1)$ has finite energy, Eq. (5), if and only if satisfies the limit conditions

$$\lim_{s\to-\infty} u(s) = p, \quad \text{and} \quad \lim_{s\to+\infty} u(s) = q$$

for some $p,q \in L_0 \cap L_1$. For the details see J. Robbin and D. Salamon [36].

Note that the strip $\mathbb{R}\times[0,1]i \subset \mathbb{C}$ is conformally equivalent with the closed unit disk $D^2 \subset \mathbb{C}$ minus two points on the boundary. Thus sometimes u is also referred as a holomorphic disk.

For $p,q \in L_0 \cap L_1$, let $C^\infty(\mathbb{R}\times[0,1],M;L_0,L_1)$ be the set of smooth maps $u: \mathbb{R}\times[0,1] \to M$ with the limit behavior as in (3). Thus we have a bundle map

$$\cup_u C^\infty(u^*(TM)) \to C^\infty(\mathbb{R}\times[0,1],M;L_0,L_1),$$

where the space $C^\infty(u^*(TM))$ is the space of vector fields along $u(\mathbb{R}\times[0,1])$, that is sections of $u^*(TM) \to \mathbb{R}\times[0,1]$. Notice that the Cauchy-Riemann equation (4) defines a section, $u \mapsto \overline{\partial}_J(u)$, of this bundle. Moreover the moduli space $\hat{\mathscr{M}}_J(p,q,L_0,L_1)$ is precisely the zero locus of this section. In order to show that the moduli space is a finite dimensional manifold, the section $\overline{\partial}_J$ must intersect transversally the zero-section. Transversality is one of the problems in defining Lagrangian Floer homology, it is a delicate issue of the subject.

The issue of transversality of the section $\overline{\partial}_J$ is in fact relaxed, from the one stated above in the sense that the smooth condition on the map u is relaxed. The smooth condition is weakened to the Sobolev space $W^k_p(\mathbb{R}\times[0,1],M;L_0,L_1)$ for $k > p/2$ and $p > 1$. However, the new zero locus obtained in this setting coincides with the previous one due to elliptic regularity; that is if $u \in W^k_p(\mathbb{R}\times[0,1],M;L_0,L_1)$ is such that $\overline{\partial}_J(u) = 0$, the u is in fact smooth.

The main result in this direction is that there exists a dense subset $\mathscr{J}_{\text{reg}}(L_0,L_1)$ of $C^\infty([0,1],\mathscr{J})$ of ω-compatible almost complex structures such that for $J = \{J_t\} \in \mathscr{J}_{\text{reg}}(L_0,L_1)$ and every $u \in \hat{\mathscr{M}}_J(p,q,L_0,L_1)$ the linearized operator

$$D(\overline{\partial}_J)_u : \{\xi \in W^k_p(u^*(TM)) | \xi(s,0) \in L_0 \text{ and } \xi(s,1) \in L_1\} \to W^{k-1}_p(u^*(TM))$$

is a surjective Fredholm operator. Furthermore the index of the operator $D(\overline{\partial}_J)_u$ is the Maslov index of the map $u: \mathbb{R}\times[0,1] \to (M,\omega)$. Below we give the definition of the Maslov index of u. It then follows that the kernel of $D(\overline{\partial}_J)_u$ is finite dimensional and is identified with the tangent space of $\hat{\mathscr{M}}_J(p,q,L_0,L_1)$ at u. For the details of these assertions see [10].

As in the finite dimensional Morse theory case, the space of flow lines $\hat{\mathscr{M}}_J(p,q,L_0,L_1)$ admits an action of $\mathbb{R}$ on the s coordinate. The quotient space by this action is denoted by $\mathscr{M}_J(p,q,L_0,L_1)$. The space $\mathscr{M}_J(p,q,L_0,L_1)$ still needs to be taken further apart; namely the homotopy class of an element needs to be taken into consideration. Let $\beta \in \pi_2(M, L_0 \cup L_1)$, and define $\mathscr{M}_J(p,q,L_0,L_1;\beta)$ as the elements $u \in \mathscr{M}_J(p,q,L_0,L_1)$ such that $[u] = \beta$.

In order to address the dimension of $\mathscr{M}_J(p,q,L_0,L_1)$, for the moment consider only one Lagrangian submanifold L of (M,ω). Then to each smooth map $u : (D^2, \partial D^2) \to (M,L)$, one gets a trivial fibration $u^*(TM) \to D^2$ that is symplectic. Furthermore, the fibration is trivial as symplectic bundles, $u^*(TM) \simeq \mathbb{R}^{2n} \times D^2$. Then when the fibration is restricted to ∂D^2, it defines a loop of Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$. That is, if $\Lambda(\mathbb{R}^{2n})$ represents the Grassmannian of Lagrangian subspaces then the trivialized fibration induces a map $u_L : S^1 \to \Lambda(\mathbb{R}^{2n})$. The *Maslov index* $\mu_L(u)$, of $u : (D^2, \partial D^2) \to (M,L)$ is defined to be the integer $(u_L)_*(1) \in \pi_1(\Lambda(\mathbb{R}^{2n})) \simeq \mathbb{Z}$. The Maslov index of u is well defined, it does not depend on the symplectic trivialization; furthermore it only depends on the homotopy type of u relative to L. Hence the Maslov index induces a group morphism $\mu_L : \pi_2(M,L) \to \mathbb{Z}$. The above concept extends to the case when two Lagrangian submanifolds L_0 and L_1 are involved. For the definition of the Maslov index see [2], [23] or [36].

Theorem 9. *Let L_0 and L_1 be compact Lagrangian submanifolds of (M,ω) that intersect transversally. Then there exists a dense subset $\mathscr{J}_{\text{reg}}(L_0,L_1)$ in $C^\infty([0,1],\mathscr{J})$ of ω-compatible almost complex structures, such that for $J = \{J_t\} \in \mathscr{J}_{\text{reg}}(L_0,L_1)$, p and q in $L_0 \cap L_1$, and $\beta \in \pi_2(M, L_0 \cup L_1)$, the space $\hat{\mathscr{M}}_J(p,q,L_0,L_1;\beta)$ is a smooth manifold. Moreover its dimension is given by the Maslov index $\mu(\beta)$.*

The proof of this result appears in [10] and [30]. So far we imposed conditions on L_0, L_1 and (M,ω), in order to have a more transparent exposition of the subject. All the statements made so far hold for arbitrary closed symplectic manifolds and compact Lagrangian submanifolds, with the corresponding adaptations. However the next results that we are going to state does not hold in general. In fact it is well understood that Lagrangian Floer homology can not be defined on arbitrary symplectic manifolds for arbitrary Lagrangian submanifolds.

See [14] and [30] for more information on this peculiarity.

For instance one can impose the condition that the pair of Lagrangians submanifolds of (M,ω) must be monotone and that the Maslov index of the Lagrangians has to be greater than 2. A Lagrangian submanifold $L \subset (M,\omega)$ is called *monotone* if there exists $\lambda > 0$, such that $\omega = \lambda \mu_L$. One important fact that follows by considering monotone Lagrangian submanifold, is that the Maslov index of a non constant holomorphic disk with boundary in the Lagrangian is positive. Under these conditions, Y.-G. Oh [30] defined Lagrangian Floer homology.

There are less restrictive conditions for which Lagrangian Floer homology is well defined; see [14] and [40]. The advantage of the monotone assumptions is that the complex is just the $\mathbb{Z}_2$-vector space generated by the intersection points of the Lagrangian submanifolds; and the sum in the definition of the boundary operator (6) is a finite sum. Basically the same picture as in the Morse theory case.

From now we are going to assume that the symplectic area of any 2-disk with boundary in the Lagrangian submanifold is zero. *Thus from now on we assume that*

(M,ω) is a closed symplectic manifold and L_0, L_1 are closed Lagrangian submanifolds such that $[\omega]\cdot\pi_2(M,L_j)=0$ for $j=0,1$. Under this conditions we will define the Floer complex of the Lagrangian submanifolds L_0 and L_1.

Then under these hypothesis, when $\mu(p,q,\beta)=1$ the space $\mathscr{M}_J(p,q,L_0,L_1;\beta)$ is compact and hence is a finite collection of points. The reason why the space is compact is because $[\omega]\cdot\pi_2(M,L_j)=0$ rules out the existence of holomorphic spheres and disks with boundary in the Lagrangian submanifolds. Since a convergent sequence of holomorphic strips, under Gromov's topology, converges to a holomorphic strip with the possible union of holomorphic spheres and disks, the moduli space is compact. See for example [30]. In some sence, this is most straightforward way scenario to define Lagrangian homology; avoid holomorphic spheres and disks.

Now the advantage of using the field $\mathbb{Z}_2$, is that we don't have to worry about orientations of the moduli spaces, that is assigning $(+)$ or $(-)$ to each component of $\mathscr{M}_J(p,q,L_0,L_1;\beta)$ Recall that $J=\{J_t\}$ is given as in Theorem 9. Denote by $\#_{\mathbb{Z}_2}\mathscr{M}_J(p,q,L_0,L_1;\beta)$ the number of points of $\mathscr{M}_J(p,q,L_0,L_1;\beta)$ module 2. Before defining the boundary operator as before, we note that the solutions of (4) might determine an infinite number of homotopy classes of $\pi_2(M,L_0\cup L_1)$. Thus we introduce the Novikov field over $\mathbb{Z}_2$ to give meaning to the possible infinite number of homotopy classes of connecting orbits. The *Novikov field* is defined as

$$\Lambda := \left\{ \sum_{j=0}^{\infty} a_j T^{c_j} | a_j \in \mathbb{Z}_2, c_j \in \mathbb{R}, \lim_{j\to\infty} c_j = \infty \right\}.$$

Let $\mathrm{CF}(L_0,L_1)$ be the free Λ-module generated by the intersection points $L_0\cap L_1$, which are finitely many since L_0 and L_1 are compact and intersect transversally. Then for $p\in L_0\cap L_1$ and $J=\{J_t\}$ as in Theorem 9, the boundary operator is defined as

$$\partial_J(p) := \sum_{\substack{q\,\in\, L_0\cap L_1,\,\beta\,\in\,\pi_2(M,L_0\cup L_1)\\ \mu(p,q,\beta)\,=\,1}} \#_{\mathbb{Z}_2}\mathscr{M}_J(p,q,L_0,L_1;\beta)\; T^{\omega(\beta)}\, q. \tag{6}$$

Theorem 10. *Let L_0 and L_1 be closed Lagrangian submanifolds of (M,ω) that intersect transversally with (M,ω) also closed and $\{J_t\}$ an almost complex structure given by Theorem 9. If $[\omega]\cdot\pi_2(M,L_j)=0$ for $j\in\{0,1\}$, then the boundary operator $\partial_J:\mathrm{CF}(L_0,L_1)\to\mathrm{CF}(L_0,L_1)$ satisfies*

$$\partial_J\circ\partial_J=0. \tag{7}$$

For the proof of this result see [10]. For the case of monotone Lagrangian submanifolds see [30].

Is important to know that the condition $[\omega]\cdot\pi_2(M,L_j)=0$ is fundamental in order to have $\partial_J\circ\partial_J=0$. In general Eq. (7) does not hold for arbitrary closed symplectic manifolds (M,ω) and compact Lagrangian submanifolds L_j. In the context of Theorem 10, the complex $(\mathrm{CF}(L_0,L_1),\partial_J)$ is called the *Floer chain complex* of (L_0,L_1). The *Lagrangian Floer Homology of* (L_0,L_1) is defined to be the homology this complex,

$$\mathrm{HF}(L_0, L_1, J) := \frac{\mathrm{Ker}\ \partial_J}{\mathrm{Im}\ \partial_J}.$$

The definition of the Floer differential is similar to the one in Morse theory; in particular in both cases we are counting the number of trajectories module 2. As pointed out before, it is possible to use integer coefficients in the case of Morse homology. This is possible by fixing a coherent system of orientation on each moduli space of gradient trajectories. However in the context of Lagrangian Floer homology this is not the case; the orientation issue is more involved in the Floer case.

Theorem 10 is the cornerstone of Lagrangian Floer homology. Here the hypothesis that the symplectic area of any 2-disk with boundary in an Lagrangian submanifold is crucial. In fact there are known examples where Theorem 10 fails; [30], [14, Ch. 2]. The next result, Theorem 11, is the philosophy of Lagrangian Floer homology. Namely, is a blueprint to solve Arnol'd Conjecture that we will see below and in Section 8.

Theorem 11. *Let L_0, L_1 and (M, ω) as in Theorem 10. Then*

(a) $\mathrm{HF}(L_0, L_1, J)$ *is independent of* $J = \{J_t\} \in \mathscr{J}_{\mathrm{reg}}(L_0, L_1)$,
(b) $\mathrm{HF}(L_0, L_1) \simeq \mathrm{HF}(L_0, \phi(L_1))$ *where ϕ is any Hamiltonian such that L_0 and $\phi(L_1)$ intersect transversally, and*
(c) $\mathrm{HF}(L, L) \simeq H_*(L, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Lambda$.

The isomorphisms in (b) and (c) are as Λ-modules.

In (c), $\mathrm{HF}(L, L)$ is understood in the sense of (b). That is, $\mathrm{HF}(L, L)$ is defined as $\mathrm{HF}(L, \phi(L))$ where ϕ is any Hamiltonian diffeomorphisms such that L and $\phi(L)$ intersect transversally.

Theorems 10 and 11 reflect the idea of what is expected of Lagrangian Floer homology theory; in the sense that the theory must solve Arnold's conjecture. First, one requires the differential complex to be generated by the intersection points of the Lagrangian submanifolds L_0 and L_1, and the differential operator to square to zero. Thus one has a homology theory, $\mathrm{HF}(L_0, L_1)$, of the pair of Lagrangian submanifolds (L_0, L_1). Finally one expects the theory to satisfy (b) and (c) of Theorem 11. With this at hand, we have for any Lagrangian submanifold L and any Hamiltonian diffeomorphism ϕ, such that L and $\phi(L)$ intersect transversally, that

$$\begin{aligned}
\#(L \cap \phi(L)) &\geq \dim_\Lambda \mathrm{HF}(L, \phi(L)) \\
&= \dim_\Lambda \mathrm{HF}(L, L) \\
&= \dim_\Lambda \Lambda \otimes_{\mathbb{Z}_2} H_*(L, \mathbb{Z}_2) \\
&= \sum_{j=0}^{n} \mathrm{Rank}\ H_j(L, \mathbb{Z}_2).
\end{aligned}$$

where the Lagrangian L has dimension n.

Theorem 12 (A. Floer [10]). *Let (M,ω) be a closed symplectic manifold, L a Lagrangian submanifold and ϕ a Hamiltonian diffeomorphisms such that L and $\phi(L)$ intersect transversally. Further, assume that $[\omega]\cdot\pi_2(M,L)=0$, then*

$$\#(L\cap\phi(L)) \geq \sum_{j=0}^{n} \text{Rank } H_j(L,\mathbb{Z}_2).$$

This result was generalized by K. Fukaya, Y.-G. Oh, H. Otha and K. Ono in [14] using the technique of Kuranishi structures on the moduli space. The hypothesis $[\omega]\cdot\pi_2(M,L)=0$ is replaced by the requirement that the map $H_*(L,\mathbb{Q})\to H_*(M,\mathbb{Q})$ must be injective. Notice that this new hypothesis can not be relaxed. As mentioned in Section 4, any symplectic manifold admits arbitrary small Lagrangian tori that are displaceable.

Now in the context of Arnol'd Conjecture, let (M,ω) be a closed symplectic manifold and ψ be a Hamiltonian diffeomorphism with non-degenerate fixed points. Then we know that graph(ψ) is a Lagrangian submanifold of the symplectic manifold $(M\times M, \pi_1^*(\omega)-\pi_2^*(\omega))$. Since the fixed points of ψ are non-degenerate, the intersection of graph(ψ) with the diagonal Δ is transversal. Now assume that $(M\times M,\pi_1^*(\omega)-\pi_2^*(\omega))$ and graph(ψ) satisfy the hypotheses about the zero symplectic area of any 2-disk with boundary in the Lagrangian, then the above computation implies that

$$\begin{aligned}\#(\Delta\cap\text{graph}(\psi)) &= \#(\Delta\cap(1\times\psi)(\Delta))\\ &\geq \sum_{j=0}^{2n}\text{Rank } H_j(\Delta,\mathbb{Z}_2)\\ &= \sum_{j=0}^{2n}\text{Rank } H_j(M,\mathbb{Z}_2).\end{aligned}$$

That is $|\text{Fix}(\psi)|\geq\sum_{j=0}^{2n}\text{Rank } H_j(M,\mathbb{Z}_2)$.

Corollary 2 (A. Floer [10]). *Let (M,ω) be a closed symplectic manifold and ϕ a Hamiltonian diffeomorphism with non-degenerate critical points. If $\pi_2(M)=0$, then*

$$|\text{Fix}(\phi)| \geq \sum_{j=0}^{2n}\text{Rank } H_j(M,\mathbb{Z}_2).$$

These notes are far from been an introduction to the subject of Lagrangian Floer homology. There are many important issues that we did not mentioned at all. For example the regularity of holomorphic strip, and the transversality issue to assure that the space of trajectories $\hat{\mathscr{M}}(p,q;\beta)$ is a smooth manifold. Another important fact that we left out was the compactness issue of $\hat{\mathscr{M}}(p,q;\beta)$, among other things. See [3], [14] and [40]. Finally we would like to mention that Lagrangian Floer homology admits a product structure

$$\mathrm{HF}(L_0,L_1)\otimes\mathrm{HF}(L_1,L_2)\to\mathrm{HF}(L_0,L_2).$$

That is, instead of considering only two Lagrangian submanifolds one considers three Lagrangians. Instead of considering holomorphic strips, one considers holomorphic triangles, that is map from the closed unit disk minus three point of the boundary; and each component of the boundary is mapped to a different Lagrangian submanifold. Of course there is nothing special of taking three Lagrangian submanifolds, one can consider any number of Lagrangian submanifolds and the corresponding holomorphic polygons. In formal terms, is said that Lagrangian Floer homology admits an A_∞-structure. For further reading of this structure see [14] and [40]. Also [4] for an introduction into the subject.

7 Computation of HF(L,L)

In this section we give the outline that shows that $\mathrm{HF}(L,L)=\mathrm{HF}(L,\phi(L))$ is isomorphic to $H_*(L,\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\Lambda$ under the assumption that $[\omega]\cdot\pi_2(M,L)=0$. As pointed out in the previous section, this isomorphism is fundamental in the proof of Arnol'd Conjecture. The idea behind the proof of $\mathrm{HF}(L,\phi(L))\simeq H_*(L,\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\Lambda$ is to prove it in the case when the symplectic manifold is $(T^*L,-d\lambda_{\mathrm{can}})$ and the Lagrangian submanifold is the zero section L_0. That is $\mathrm{HF}(L_0,\phi(L_0))\simeq H_*(L_0,\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\Lambda$ for some particular Hamiltonian $\phi: T^*L\to T^*L$. Though in this case the symplectic manifold is non compact, together with the hypothesis $[\omega_{\mathrm{can}}]\cdot\pi_2(T^*L,L_0)=0$ we assume that Theorem 10 still hols in this case.

The proof of the general case, that is for an arbitrary compact symplectic manifold (M,ω) and a Lagrangian submanifold L relies on Weinstein's Lagrangian tubular neighborhood theorem. That is, consider a Hamiltonian $\phi: M\to M$ such that $\phi(L)$ lies in a Weinstein's tubular neighborhood of L. Then under the symplectic diffeomorphism given by Weinstein theorem, the known computations of $(T^*L,-d\lambda_{\mathrm{can}})$, that is holomorphic disks and gradient flow lines in L, are carried to (M,ω) and L to obtain the isomorphism between $\mathrm{HF}(L,\phi(L))$ and $H_*(L,\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\Lambda$.

Now we show the Riemannian structure and almost complex structure on the cotangent bundle that will be relevant in this section in the computation of Lagrangian Floer homology. Hence given a Riemannian metric g on L it induces an almost complex structure $\mathbf{J}$ on T^*L that is compatible with ω_{can}. Such that at the zero section $(p,0)\in T^*L$, it maps vertical vectors $T^*_pL\subset T_{(p,0)}(T^*L)$ to horizontal vectors $T_pL\subset T_{(p,0)}(T^*L)$. Set $\mathbf{g}$ to be the Riemannian structure induced by $\mathbf{J}$ and ω_{can}, thus $\mathbf{g}(\cdot,\cdot)=\omega_{\mathrm{can}}(\cdot,\mathbf{J}\cdot)$. Notice that such almost complex structure $\mathbf{J}$ is not unique.

Now lets see the relation between Morse homology on L and Lagrangian Floer homology of the zero section L_0 in $(T^*L,\omega_{\mathrm{can}})$. To that end let g be a Riemannian structure on L and $f: L\to\mathbb{R}$ be a Morse-Smale function. Then the graph of the 1-form df,

$$L_1 := \mathrm{graph}(df)=\{(p,df_p)|p\in L\},$$

is a Lagrangian submanifold in $(T^*L, \omega_{\text{can}})$ since is the graph of a closed 1-form. Therefore L_1 intersects L_0 precisely at the the critical points of f; that is at points $(p,0)$ for $p \in \text{Crit}(f)$. Furthermore the intersection is transversal and consists of finitely many points.

As before let $\pi : T^*L \to L$ be the projection map, set $H := -f \circ \pi$ and X_H the Hamiltonian vector field of H on $(T^*L, \omega_{\text{can}})$, thus

$$\omega_{\text{can}}(X_H, \cdot) = dH.$$

Notice that if $(x_1, \ldots, x_n)$ are the local coordinates of L, and $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ the corresponding local coordinantes of T^*L as in Example 6, then the Hamiltonian vector field X_H takes the form,

$$X_H := \frac{\partial f}{\partial x_1}\frac{\partial}{\partial y_1} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial}{\partial y_n}.$$

This expression justifies the minus sign in the definition of the Hamiltonian function H. Finally let $\phi_t : T^*L \to T^*L$ be the path of Hamiltonian diffeomorphisms induced by the Hamiltonian function H.

In local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, the path $\{\phi_t\}$ of Hamiltonian diffeomorphisms takes the form

$$\phi_t(x_1, \ldots, x_n, y_1, \ldots, y_n) = \left(x_1, \ldots, x_n, t\frac{\partial f}{\partial x_1}(x) + y_1, \ldots, t\frac{\partial f}{\partial x_n}(x) + y_n \right).$$

Therefore $\phi_1(p,0) = (p, df_p)$.

Proposition 2. *Let $\phi_t : T^*L \to T^*L$, L_0 and L_1 as above. Then the time-1 map is such that $\phi_1(L_0) = L_1$.*

However to describe X_H in global terms we must use the Riemannian structure g on L and the remark made above. Thus let $\mathbf{J}$ on $(T^*L, \omega_{\text{can}})$ as before subject to the condition that on the zero section $\mathbf{J}\,\text{grad}(f) = df \in T^*L \subset T(T^*L)$ where $\text{grad}(f) \in TL \subset T(T^*L)$. Furthermore on the zero section we have that

$$\mathbf{grad}(H) = \text{grad}(f).$$

Since $\mathbf{J}$ is compatible with ω_{can} it follows from $\omega_{\text{can}}(X_H, \cdot) = dH$ that $\mathbf{grad}(H) = -\mathbf{J}X_H$ and $X_H = \mathbf{J}\,\text{grad}(f)$.

In the context of Lagrangian Floer homology we care about the intersection points of L_0 with $\phi(L_0)$, that in this particular case are in a natural bijection with the critical points of f. Therefore there is a natural Λ-linear bijection

$$I : \text{CF}(L_0, \phi(L_0)) \to \text{Crit}(f) \otimes_{\mathbb{Z}_2} \Lambda$$

which on generators takes the form $I(p,0) = p$. This map will give rise to the isomorphism between $\text{HF}(L_0, \phi(L_0))$ and $H_*(L, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Lambda$. It remains to look at the differentials maps in each case; and henceforth holomorphic disks and gradient flow lines.

However in order to relate the flow lines in L of the gradient of f of Morse index 1 that come into play in the Morse differential, with the holomorphic disks in T^*L with boundary in L_0 and and of Maslov index 1 that come into play in the Floer differential, and additional condition on f is imposed. Namely f must be C^2-small with respect to g. Recall that we assume that $f : N \to \mathbb{R}$ satisfies the Morse-Smale condition. Then according to A. Floer [11], the family of almost complex structures $J_t := -\phi_{t,*} \circ \mathbf{J} \circ (\phi_{t,*})^{-1}$ for $0 \leq t \leq 1$, is regular. That is $J = \{J_t\} \in \mathscr{J}_{\mathrm{reg}}(L_0, L_1)$. Since we assume that $[\omega_{\mathrm{can}}] \cdot \pi_2(T^*L, L_0) = 0$, then the Lagrangian Floer homology of (L_0, L_1) can be computed using the moduli spaces $\mathscr{M}_J(p,q,L_0,L_1)$, where p and q are intersection points. Recall that for p and q critical points of $f : L \to \mathbb{R}$, $\mathscr{M}(f;p,q)$ is the moduli space of flow lines of $-\mathrm{grad}(f)$ connecting p to q.

Consider the map

$$\Psi : \mathscr{M}(f;p,q) \to \mathscr{M}_J(p,q,L_0,L_1)$$

given by $\Psi(u)(s,t) := \phi_t(u(s))$. Next we show that Ψ is well defined, that is that $\phi_t(u(s))$ is J-holomorphic and satisfies the boundary conditions. For note that $\Psi(u)(s,0) = \phi_0(u(s))$ lies in L_0 and $\Psi(u)(s,1) = \phi_1(u(s))$ lies in L_1. It only remains to show that $\Psi(u)$ is J-holomorphic. To that end notice that

$$\frac{du}{dt}(t) = -\mathrm{grad}(f)(u(t)),$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial s}\Psi(u)(s,t) &= \frac{\partial}{\partial s}\phi_t(u(s)) = \phi_{t,*}(u(s))\left(\frac{d}{ds}u(s)\right)\\ &= -\phi_{t,*}(u(s))\left(\mathrm{grad}(f)(u(t))\right).\end{aligned}$$

On the other hand, since $X_H = \mathbf{J}\,\mathrm{grad}(f)$ and $-J_t \circ \phi_{t,*} = \phi_{t,*} \circ \mathbf{J}$ it follows that

$$\begin{aligned}\frac{\partial}{\partial t}\Psi(u)(s,t) &= \frac{\partial}{\partial t}\phi_t(u(s)) = X_H(\phi_t(u(s)))\\ &= \phi_{t,*}(u(s))(X_H(u(s)))\\ &= \phi_{t,*}(u(s))(\mathbf{J}\ \mathrm{grad}(f)(u(s)))\\ &= -J_t\phi_{t,*}(u(s))(\mathrm{grad}(f)(u(s)))\end{aligned}$$

Since $J_t^2 = -1$, we get that

$$\frac{\partial}{\partial s}\Psi(u)(s,t) + J_t\frac{\partial}{\partial t}\Psi(u)(s,t) = 0. \tag{8}$$

That is if u is a gradient flow line we have that $\Psi(u)$ satisfies the Cauchy-Riemann equation with respecto to $J = \{J_t\}$. That is, $\Psi : \mathscr{M}(f;p,q) \to \mathscr{M}_J(p,q,L_0,L_1)$ is well-defined and according to A. Floer [11] is a bijection for f small enough.

Recall that
$$I : \mathrm{CF}(L_0, \phi(L_0)) \to \mathrm{Crit}(f) \otimes_{\mathbb{Z}_2} \Lambda$$
given by $I(p,0) = p$ is also a bijection. Furthermore, the fact that Ψ is a bijection implies that I is a chain map, $I \circ \partial_J = \partial \circ I$. (Here the Morse differential on $\mathrm{Crit}(f)$ is extended Λ-linearly to $\mathrm{Crit}(f) \otimes_{\mathbb{Z}_2} \Lambda$) Moreover the induced map $I : \mathrm{HF}(L_0, \phi(L_0)) \to \mathrm{MH}_*(L; f, g) \otimes_{\mathbb{Z}_2} \Lambda$ is an isomorphism of Λ-modules. Finally, since $\mathrm{MH}_*(L; f, g) \simeq H_*(L, \mathbb{Z}_2)$ we have that
$$\mathrm{HF}(L_0, \phi(L_0)) \simeq H_*(L, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Lambda. \tag{9}$$
in the case of the symplectic conagent bundle $(T^*L, \omega_{\mathrm{can}})$ and the zero section L_0 Lagrangian submanifold.

The proof of the above statement in the case of an arbitrary compact symplectic manifold (M, ω) and a compact Lagrangian submanifold L, under the assumption that $[\omega] \cdot \pi_2(M, L) = 0$, is based on the on the contagent bundle case. For, consider $\phi : M \to M$ a Hamiltonian diffeomorphisms such that L and $\phi(L)$ intersect transversally and small enough so that $\phi(L)$ lies in tubular neighborhood of L. Then by Weinstein's Lagrangian neighborhood theorem, this small neighborhood of L is symplectomorphic to a neighborhood to the zero section of (T^*L, ω_{can}). Hence the all the relevant information in the computation of $\mathrm{HF}(L, \phi(L))$ lie in the tubular neighborhood of L. Thus the isomorphism (9) also holds in this case.

8 Applications

In principle Lagrangian Floer homology, was meant to solve Arnol'd conjecture. Nowadays it is important on its own. In this section we will briefly explain a few examples.

The 2-sphere example revisited. Consider (S^2, ω) and L any embedded circle. Clearly this example does not satisfy the condition $[\omega] \cdot \pi_2(S^2, L) = 0$; nevertheless for some particular Lagrangian submanifolds it lies in the monotone case. For instance if L is an equator then is a monotone Lagrangian and the Lagrangian Floer homology applies. Consider ϕ is a rotation of (S^2, ω) such that L and $\phi(L)$ are transversal, then the standard complex structure J is regular. In this case $\mathrm{CF}(L, \psi(L))$ has two generators and ∂_J is the zero map. Therefore $\mathrm{HF}(L, \phi(L))$ has rank two. As we explain above, this means that the equator is a non displaceable Lagrangian of (S^2, ω).

More generally, consider the Lagrangian submanifold $\mathbb{R}P^n$ in $(\mathbb{C}P^n, \omega_{FS})$ for $n \geq 1$. The Lagrangian $\mathbb{R}P^n$ is a monotone. On $(\mathbb{C}P^n, \omega_{FS})$ there is a canonical $SU(n+1)$ Hamiltonian action, that is induced from the standard linear action on the euclidean space $\mathbb{C}^{n+1}$. Restrict the action to the maximal torus of $SU(n+1)$. Then a vector on the Lie algebra of the maximal torus induces, by the exponential map, a Hamiltonian diffeomorphism ϕ of $(\mathbb{C}P^n, \omega_{FS})$. Moreover, is such that $\phi(\mathbb{R}P^n)$ meets $\mathbb{R}P^n$ transversally. In this example the standard complex structure

J, $\mathbb{R}P^n$ and $\phi(\mathbb{R}P^n)$ satisfy the transversality condition, that is the moduli spaces $\mathscr{M}_J(p,q;\beta)$ are smooth manifolds for $p,q \in \mathbb{R}P^n \cap \phi(\mathbb{R}P^n)$. Moreover the analog of Theorem 10 in the monotone case also holds. Hence the Lagrangian Floer homology of $(\mathbb{R}P^n, \phi(\mathbb{R}P^n))$ is well defined, further the differential ∂_J is the zero map and

$$\mathrm{HF}(\mathbb{R}P^n, \phi(\mathbb{R}P^n))$$

has rank $n+1$ as a Λ-module. In particular, $\mathbb{R}P^n$ is non displaceable. This result is due to Y.-G. Oh [31], in the setting of Lagrangian Floer homology for monotone Lagrangians.

Another important example concerns symplectic toric manifolds. In this case if (M,ω) is a toric manifold with moment map $\Phi : M \to \Delta$, then for x in the interior of the polytope Δ, $L_x = \Phi^{-1}(x)$ is a Lagrangian n-torus. In this case there is a dichotomy,

$$\mathrm{HF}(L_x, L_x) = 0 \quad \text{or} \quad \mathrm{HF}(L_x, L_x) = H_*(\mathbb{T}^n; \mathbb{Z}_2) \otimes \Lambda.$$

In fact a stronger result is true, those Lagrangian torus L_x such that $\mathrm{HF}(L_x, L_x) = 0$ are in fact displaceable.

Example 2 of the 2-sphere with a Morse function f, is an example of a symplectic toric manifold. In this case the Morse function is in fact a moment map, recall that $f : S^2 \to [0,1]$ is given by $f(x,y,z) = z$. Then for $z \in [0,1]$ not equal to zero L_z is a circle that lies entirely in the north or south hemisphere; hence L_z is displaceable. And for the equator L_0, we have

$$\mathrm{HF}(L_0, L_0) = H_*(S^1; \mathbb{Z}_2) \otimes \Lambda,$$

as mentioned at the beginning of this section. For further details on Lagrangian Floer homology on symplectic toric manifolds see [15].

References

1. ARNOL'D, V. Sur une propriété topologique des applications globalement canoniques de la mécanique classique. *C. R. Acad. Sci. Paris 261* (1965), 3719–3722.
2. ARNOL'D, V. I. *Mathematical methods of classical mechanics*, vol. 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993. Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition.
3. AUDIN, M., AND DAMIAN, M. *Morse theory and Floer homology*. Universitext. Springer, London; EDP Sciences, Les Ulis, 2014. Translated from the 2010 French original by Reinie Erné.
4. AUROUX, D. A beginners introduction to fukaya categories. In *Contact and Symplectic Topology* (2014), vol. 26 of *Bolyai Society Mathematical Studies*, Springer International Publishing, pp. 85–136.
5. BANYAGA, A. Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique. *Comment. Math. Helv. 53*, 2 (1978), 174–227.
6. BOTT, R. Morse theory indomitable. *Inst. Hautes Études Sci. Publ. Math.*, 68 (1988), 99–114 (1989).

7. CANNAS DA SILVA, A. *Lectures on symplectic geometry*, vol. 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.
8. CONLEY, C. C., AND ZEHNDER, E. The Birkhoff-Lewis fixed point theorem and a conjecture of V. I. Arnol'd. *Invent. Math. 73*, 1 (1983), 33–49.
9. ELIASHBERG, Y. M. A theorem on the structure of wave fronts and its application in symplectic topology. *Funktsional. Anal. i Prilozhen. 21*, 3 (1987), 65–72, 96.
10. FLOER, A. Morse theory for Lagrangian intersections. *J. Differential Geom. 28*, 3 (1988), 513–547.
11. FLOER, A. Witten's complex and infinite-dimensional Morse theory. *J. Differential Geom. 30*, 1 (1989), 207–221.
12. FORTUNE, B., AND WEINSTEIN, A. A symplectic fixed point theorem for complex projective spaces. *Bull. Amer. Math. Soc. (N.S.) 12*, 1 (1985), 128–130.
13. FUKAYA, K. Morse homotopy, A^∞-category, and Floer homologies. In *Proceedings of GARC Workshop on Geometry and Topology '93 (Seoul, 1993)* (1993), vol. 18 of *Lecture Notes Ser.*, Seoul Nat. Univ., Seoul, pp. 1–102.
14. FUKAYA, K., OH, Y.-G., OHTA, H., AND ONO, K. *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*, vol. 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.
15. FUKAYA, K., OH, Y.-G., OHTA, H., AND ONO, K. Lagrangian Floer theory on compact toric manifolds. I. *Duke Math. J. 151*, 1 (2010), 23–174.
16. FUKAYA, K., AND ONO, K. Arnold conjecture and Gromov-Witten invariant. *Topology 38*, 5 (1999), 933–1048.
17. GOMPF, R. E. Symplectically aspherical manifolds with nontrivial π_2. *Math. Res. Lett. 5*, 5 (1998), 599–603.
18. GROMOV, M. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math. 82*, 2 (1985), 307–347.
19. HOFER, H., AND SALAMON, D. A. Floer homology and Novikov rings. In *The Floer memorial volume*, vol. 133 of *Progr. Math.* Birkhäuser, Basel, 1995, pp. 483–524.
20. LIU, G., AND TIAN, G. Floer homology and Arnold conjecture. *J. Differential Geom. 49*, 1 (1998), 1–74.
21. MATSUMOTO, Y. *An introduction to Morse theory*, vol. 208 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2002. Translated from the 1997 Japanese original by Kiki Hudson and Masahico Saito, Iwanami Series in Modern Mathematics.
22. MCDUFF, D. Notes on kuranishi atlases. *arXiv:1411.4306*.
23. MCDUFF, D., AND SALAMON, D. *Introduction to symplectic topology*, second ed. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998.
24. MCDUFF, D., AND SALAMON, D. *J-holomorphic curves and symplectic topology*, second ed., vol. 52 of *A. M. S. Colloquium Publications*. American Mathematical Society, Providence, RI, 2012.
25. MCDUFF, D., AND WEHRHEIM, K. The fundamental class of smooth kuranishi atlases with trivial isotropy. *arXiv:1508.01560*.
26. MCDUFF, D., AND WEHRHEIM, K. Smooth kuranishi atlases with isotropy. *arXiv:1508.01556* .
27. MCDUFF, D., AND WEHRHEIM, K. The topology of kuranishi atlases. *arXiv:1508.01844*.
28. MILNOR, J. *Morse theory*. Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.
29. NICOLAESCU, L. I. *An invitation to Morse theory*. Universitext. Springer, New York, 2007.
30. OH, Y.-G. Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I. *Comm. Pure Appl. Math. 46*, 7 (1993), 949–993.
31. OH, Y.-G. Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. II. $(\mathbf{C}P^n, \mathbf{R}P^n)$. *Comm. Pure Appl. Math. 46*, 7 (1993), 995–1012.
32. OH, Y.-G. *Symplectic topology and Floer homology. Vol. 1*, vol. 28 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2015. Symplectic geometry and pseudoholomorphic curves.

33. OH, Y.-G. *Symplectic topology and Floer homology. Vol. 2*, vol. 29 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2015. Floer homology and its applications.
34. ONO, K. On the Arnol'd conjecture for weakly monotone symplectic manifolds. *Invent. Math. 119*, 3 (1995), 519–537.
35. PARDON, J. An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves. *Geom. Topol. 20*, 2 (2016), 779–1034.
36. ROBBIN, J. W., AND SALAMON, D. A. Asymptotic behaviour of holomorphic strips. *Ann. Inst. H. Poincaré Anal. Non Linéaire 18*, 5 (2001), 573–612.
37. RUAN, Y. Virtual neighborhoods and pseudo-holomorphic curves. In *Proceedings of 6th Gökova Geometry-Topology Conference* (1999), vol. 23, pp. 161–231.
38. SALAMON, D. Lectures on Floer homology. In *Symplectic geometry and topology (Park City, UT, 1997)*, vol. 7 of *IAS/Park City Math. Ser.* Amer. Math. Soc., Providence, RI, 1999, pp. 143–229.
39. SCHWARZ, M. *Morse homology*, vol. 111 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1993.
40. SEIDEL, P. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.
41. TU, L. W. *An introduction to manifolds*, second ed. Universitext. Springer, New York, 2011.
42. WITTEN, E. Supersymmetry and Morse theory. *J. Differential Geom. 17*, 4 (1982), 661–692 (1983).